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CAPABILITIES OF MULTIPLICATIVE ARRAY PROCESSORS AS SIGNAL DETECTOR AND BEARING ESTIMATOR

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Investigation is made of a "multiplicative" receiver array to determine its capability as signal detector and bearing estimator. The same analyses are made of the conventional square-law array for comparison purposes. The probability density functions for the filter outputs of array models are obtained for these arrays for the input signals consisting of monochromatic signals and narrowband additive Gaussian noise. The number of sensor elements, SNR, source bearing, and interelement noise correlation, are treated as

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18. Certain portions of the technical contents of this report, and some applications of the basic results obtained under this contract, were published in the following Journals. (Reprints or manuscripts are attached at the end of this report.)

1. "Signal Detection and Bearing Estimation by Square-Law and Multiplicative Array Processors," Proc. 1973 International Conference on Engineering in the Ocean Environment, pp. 475-480. (IEEE publication 73 CH0774-0 OCC)

2. "The Probability Density Function for the Output of an Analog Cross-Correlator with Correlated Bandpass Inputs," IEEE Transactions on Information Theory, Vol. IT-20, No. 4, July 1974, pp. 433-440.

3. "On the Binary DPSK Communication Systems in Correlated Gaussian Noise," IEEE Transactions on Communications, Vol. COM-23, No. 2, Feb. 1975.

The above papers are included at the end of this report.

20. (cont.) parameters of the distribution, and their influences are displayed graphically. Based on these probability density functions for the square-law and multiplicative array processors, we then examine their performances as signal detectors and bearing estimators.

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CHAPTER ONE

INTRODUCTION

How best to detect known signals in noise is the object of much study and effort. Among the forms in which this problem occurs is that of deciding when a signal is present and from what direction it is arriving. In this study, we are concerned with the capabilities of sonar arrays to perform these functions.

1.1 BACKGROUND OF THE PROBLEM

The emphasis to be found here is upon analytical rigor, and so the class of array models treated is restricted. However, in the presentation of this work we have striven to facilitate extensions to related problems by being quite general in the earlier stages of the analysis.

Specifically, we shall derive the detection and bearing estimation performance of a "multiplicative linear array" in comparison with that of the "standard linear array."

1.1.1 LINEAR ARRAYS

By the term "linear array" we refer to a set of 2M isotropic point receivers (hydrophones), arranged in a straight line with equal spacing d. In the literature, this configuration is often called a "broadside array," particularly in connection with transmitting systems. In receiving applications this configuration makes use of the greater coherence or correlation among the samples of the signal present at the hydrophones over those of the noise, when this condition exists. In effect, it is a method of simultaneously obtaining 2M samples or observations of the signal plus noise process, a spatial analog to time sampling.

When these samples are summed, the signal components tend to reinforce each other while the noise components, under certain conditions, tend to cancel or "average out." Also, when the signal is in the form of a sinusoidal plane wave, the individual signal components sum in such a way as to give the array a highly directive receiving pattern.

In this plane wave case, which shall be our signal model throughout the study, the signal components at the hydrophones can be modeled as $s_k(t) = A_k(t)\cos[\omega t-\emptyset-(k-1)\beta\sin\alpha]$, $k = 1,2,\ldots 2M$, where α is the bearing of the (distant) source relative to the perpendicular of the array, $\emptyset$ is an arbitrary initial phase, and $\beta = 2\pi d/\lambda$. The angular frequency is equal to $2\pi c/\lambda$, where c and λ are the velocity of sound and the wavelength in water, respectively.

For a symmetrical gain array ($A_k = A_{2M+1-k}$, $k = 1,2,\ldots M$), the sum of the signal components is

$$s(t) = \sum_{k=1}^{2M} s_k(t) = 2\cos[\omega t-\emptyset-\tfrac{1}{2}(2M-1)\beta\sin\alpha]\sum_{k=1}^{M} A_{M+1-k}\cos[\tfrac{1}{2}(2k-1)\beta\sin\alpha] \tag{1-1}$$

and for a uniform gain array ($A_k = A$, all k), the sum is

$$s(t) = 2A\cos[\omega t-\emptyset-\tfrac{1}{2}(2M-1)\beta\sin\alpha]\sum_{k=1}^{M}\cos[\tfrac{1}{2}(2k-1)\beta\sin\alpha] \tag{1-2}$$

$$= 2MAD(\tfrac{1}{2}\beta\sin\alpha;\ 2M)\cos[\omega t-\emptyset-\tfrac{1}{2}(2M-1)\beta\sin\alpha],$$

where $D(x;\ N) = \sin Nx/N\sin x$.

The sum of the noise components meanwhile, to illustrate with the simple case in which the components are independent, zero-mean, and have equal variances σ^2, would have a variance equal to $2M\sigma^2$. Thus, in this ideal case, at zero bearing the uniform linear array sum would achieve a signal to noise ratio "gain" of

$$(SNR)_{out}/(SNR)_{in} = \frac{\frac{1}{2}(4M^2A^2)}{2M\sigma^2} \div \frac{A^2}{2\sigma^2} = 2M \tag{1-3}$$

In general there exist proportional relationships between the number of elements and the array processing gain, and between the number of elements and the array directivity, considered separately. However, as will be discussed below, the relationship between directivity and SNR gain is subject to other considerations.

Schelkunoff [1], Dolph [2], and others, by considering the directivity function $D(\frac{1}{2}\beta \sin\alpha; N)$ as a polynomial in $\cos(\frac{1}{2}\beta \sin\alpha)$, have shown that the directivity of a linear array whose elements are summed can be optimized as a function of the inter-element spacings and/or the individual weightings of the elements. In this study we shall restrict our attention to uniform gain linear arrays with equal spacings.

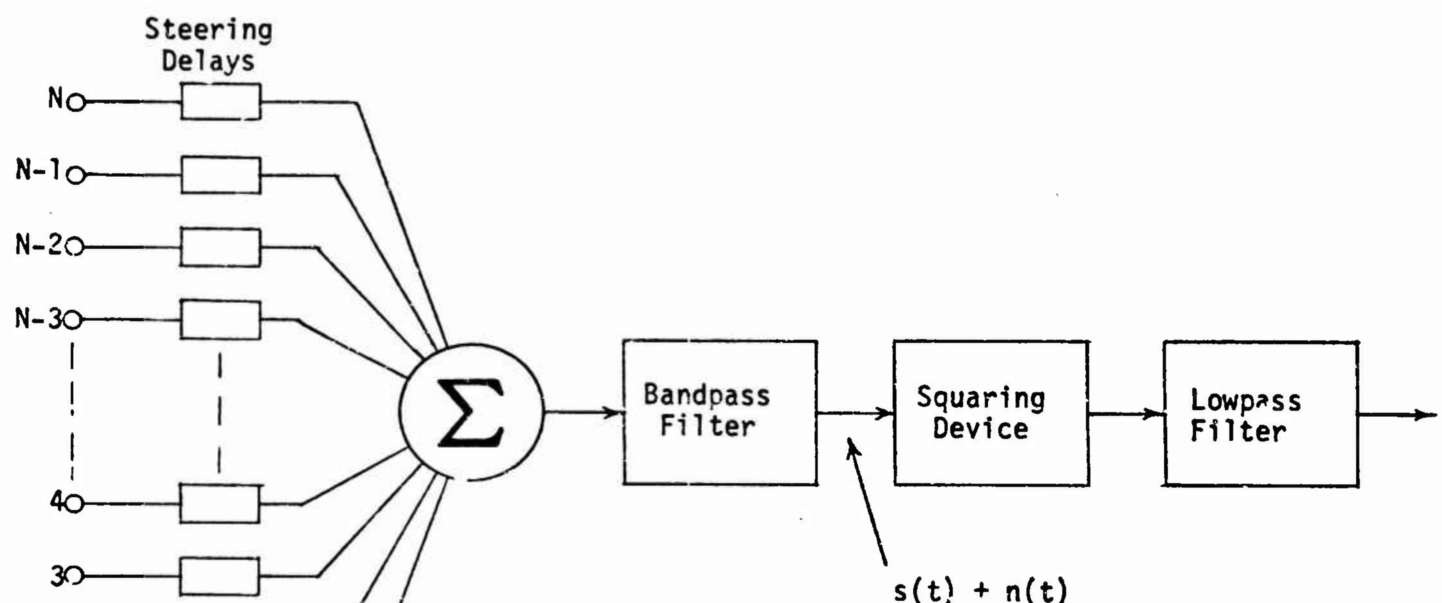
1.1.2 SQUARE-LAW ARRAY PROCESSING

In his textbook an acoustic signal processing, Horton [3] terms "standard detector" that configuration in which the sum of the hydrophone outputs is squared and averaged by means of a lowpass filter, usually with a variable time constant or bandwidth to accommodate tracking conditions. The output of the filter is then observed as the array is steered either physically or electronically. When the observation exceeds a certain threshold, the decision, "signal present," is made. When, as the steering takes place, the observation passes through a peak value, then the bearing of the signal source is taken to be the direction in which the array was steered at the time. A diagram representing the standard or square-law processor is shown in Figure 1-1.

1.1.3 MULTIPLICATIVE ARRAY PROCESSING

Berman and Clay [4] showed that methods of generating polynomials can be found which require fewer array elements than the uniform linear array whose outputs are summed. One such method was shown to be taking time averages of products of various array element signals, rather than summing them, and manipulating the relative spacings among elements. Thus a 4-element TAP (time-averaged-product) array achieved the direction pattern of an 8-element additive array, and an 11-element TAP array matched the pattern of a 1024-element additive array.

Tucker and Welsby [5-8] and Cath [9] considered systems in which the N elements of a uniform gain linear array are divided into two groups; the summed output of one group is multiplied by that of the other, and the product is smoothed by lowpass filtering--in essence, the correlation of two arrays of smaller size. The beamwidth of this configuration, shown diagrammatically in Figure 1-2, was shown to be about one half that of the corresponding N-element standard array. Also, the relative magnitudes of the unwanted sidelobes are greatly reduced, the largest sidelobes being of negative polarity and thus removable by rectification. These advantages are illustrated in Figure 1-3 for a specific case.

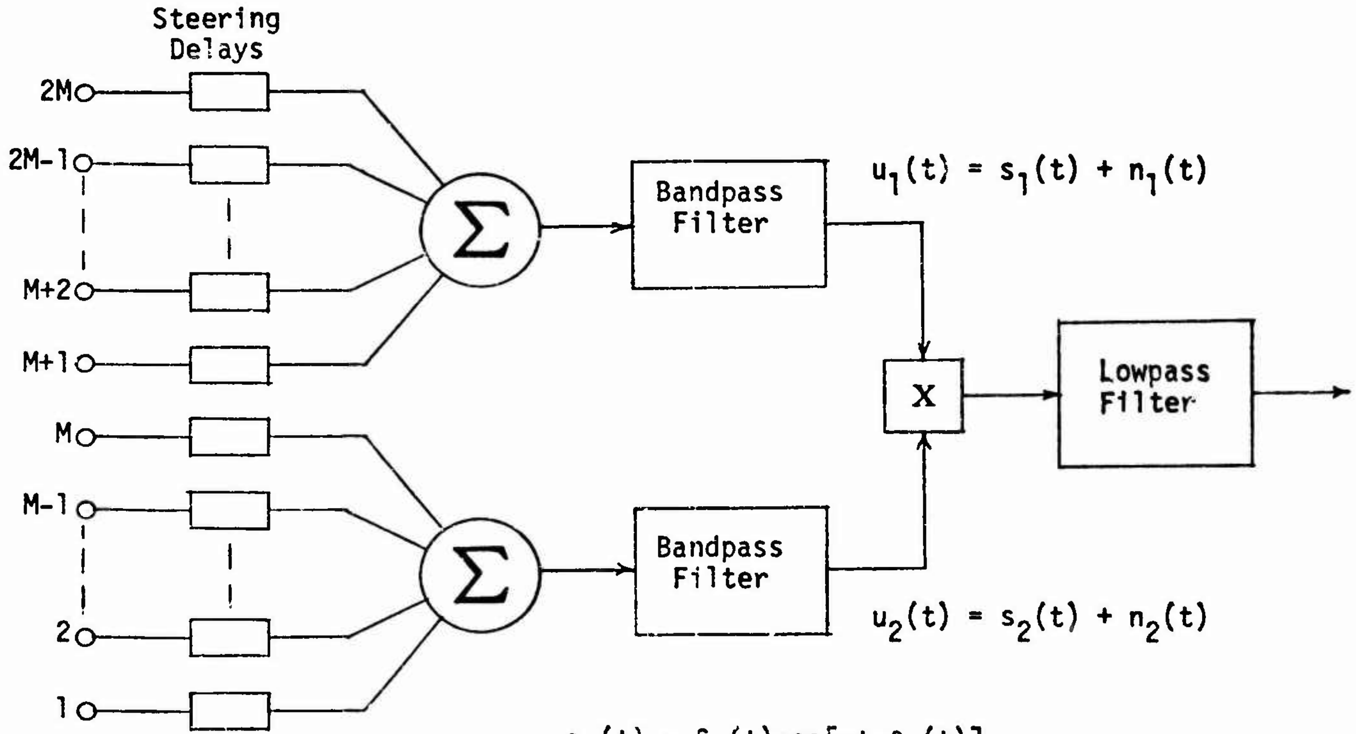


$$s(t) = S(t)\cos(\omega t-\theta)$$

$$n(t) = N(t)\cos(\omega t-\phi)$$

$$= n_c(t)\cos\omega t + n_s(t)\sin\omega t$$

FIGURE 1-1 STANDARD ARRAY PROCESSOR



$$s_i(t) = S_i(t)\cos[\omega t-\theta_i(t)]$$

$$n_i(t) = N_i(t)\cos[\omega t-\Phi_i(t)] \qquad i = 1, 2$$

$$= n_{ic}(t)\cos\omega t + n_{is}(t)\sin\omega t$$

FIGURE 1-2 MULTIPLICATIVE ARRAY PROCESSOR

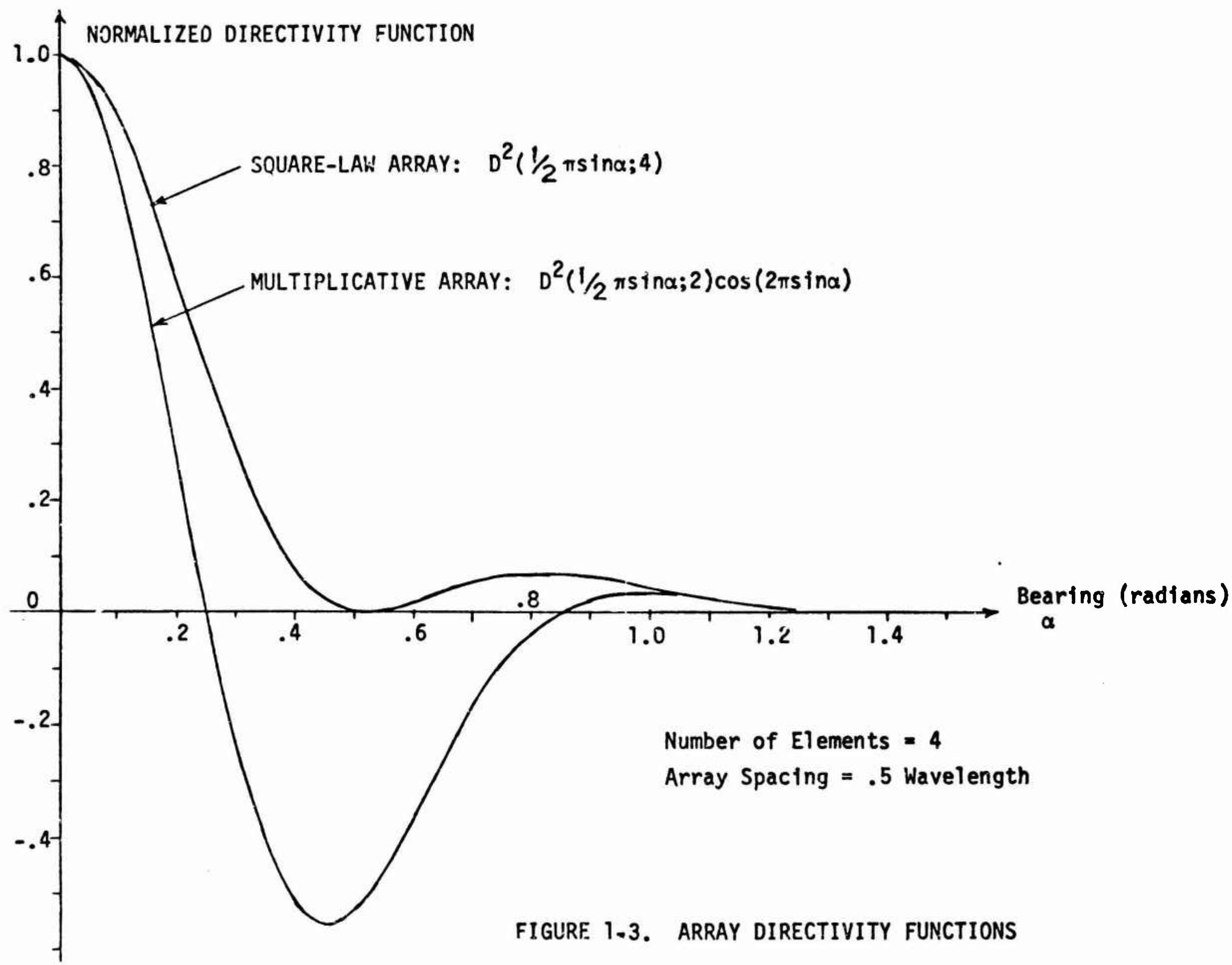


FIGURE 1-3. ARRAY DIRECTIVITY FUNCTIONS

While multiplier configurations can offer more efficient beamforming, in the processing area they represent a departure from conventional detection toward correlation techniques. Thus there come into play additional tradeoffs, earlier stated by Faran and Hills [10], and subsequently confirmed by these array designers in their calculations and experiments.

It was found that any improvement in the directional pattern over the standard array by multiplicative processing is accompanied by degradation of the noise factor, or decreased SNR gain. Also, the multiplication tends to mar the performance of discrimination between two targets or sources of unequal strength, while for equal strengths the performance was demonstrated to be greater.

The degradation in array gain for TAP arrays relative to standard was examined by Fakley [11] for narrowband noise and coherent narrowband signals. He shows that in a specific instance, for the same probability of correct detection, TAP operations require a higher input SNR. Fakley also shows the occurrence of the small signal suppression effect in the case of two sources, as well as the interesting case of a line source. In the latter he presents a calculation implying that the multiplicative process output will not track the angular energy distribution of the extended source, but rather peaks at its center bearing.

Thus, among the experimenters and researchers in the field of sonar arrays, there exists a running debate on the relative merits and tradeoffs between standard and multiplicative array processing.

1.2 DESCRIPTION OF THE PRESENT WORK

This "debate" over the relative merits of standard and multiplicative arrays has been handicapped by a significant consideration. Tucker [5] alluded to it as "... the difficult question of whether the better signal/noise performance ... is really indicative of a higher probability of detection." That is to say, the present methods of characterizing the array processing performance are inadequate. Except in a few instances the performance criteria, because of the analytical difficulty of obtaining statistical detection parameters, have been SNR gains, calculated usually for rather specialized cases such as white noise.

1.2.1 OBJECTIVES AND PROCEDURES

The object of this study is to make a positive contribution to the array processing debate by deriving exact probability density functions for the filter outputs of reasonably general narrowband models of both the standard array and the multiplicative, with the major emphasis on the latter. In doing so, we have been able to develop expressions for detection performance criteria in which appear all the various parameters which go into the tradeoffs. This material is presented in Chapters Two and Three.

Also, we have utilized these probability distributions to calculate the theoretical best bearing estimation performance (minimum variance, error) attainable due to the information contained in the array filter output signal. Forms for maximum likelihood estimators are derived and evaluated with respect to the predicted minimum variance. This work is found in Chapters Four and Five.

In Chapter Six the study is concluded by summarizing the performance of the multiplicative array model, particularly in comparison wth the standard model. Suggestions also are given for extensions of the work.

1.2.2 THE ARRAY CONFIGURATIONS TO BE STUDIED

The standard array configuration, as shown in Figure 1-1, and the multiplicative array configuration mentioned by Welsby and Tucker, given in Figure 1-2, are the models to be studied--primarily the multiplicative one. That is, the multiplicative array studied shall be considered to have a total of 2M isotropic point receivers or hydrophones equally spaced d units along a straight line and each with unity gain. The outputs of these hydrophones, indexed sequentially from one end of the array to the other, are denoted by $x_k(t)$, $k = 1,2,...,2M$. The standard array shall be considered to have N hydrophones.

In the case of the standard array, the output of the processor is the filtered version of

$$z(t) = \left[\sum_{k=1}^{N} x_k(t)\right]^2 = [NA(t)D(\tfrac{1}{2}\beta\sin\alpha;\ N)\cos(\omega t-\emptyset-\tfrac{1}{2}(N-1)\beta\sin\alpha) + n(t)]^2 \tag{1-4}$$

For the multiplicative array, the output is, before filtering,

$$z(t) = \left[\sum_{k=1}^{M} x_k(t)\right]\left[\sum_{k=M+1}^{2M} x_k(t)\right]$$

$$= \{MA(t)D(\tfrac{1}{2}\beta\sin\alpha;\ M)\cos[\omega t-\emptyset-\tfrac{1}{2}(M-1)\beta\sin\alpha] + n_1(t)\}$$

$$\times \{MA(t)D(\tfrac{1}{2}\beta\sin\alpha;\ M)\cos[\omega t-\emptyset-\tfrac{1}{2}(3M-1)\beta\sin\alpha] + n_2(t)\}. \tag{1-5}$$

The form of the signal used here and throughout the study is that of a known narrowband source whose bandwidth is centered about $\omega/2\pi$, and located at bearing sufficiently far away that the array experiences it as a plane wave. This form is emphasized diagrammatically in Figures 1-1 and 1-2 by the presence of narrowband filters. For the sake of avoiding ambiguities, our attention will be mainly on bearings which fall within the major lobe of the array. That is, $|N\beta \sin \alpha| < \pi$ for the standard array and $|M\beta \sin \alpha| < \pi$ for the multiplicative. Any steering done by the array shall be assumed ideal, that is unity gain, distortionless, time delays.*

1.2.3 THE NOISE PROCESS MODEL

Although the conditions do not always warrant it, for convenience the noise process in the locality of the array shall be considered stationary Gaussian with zero mean. In Appendix A a discussion is presented to motivate the selection of values for the elements of the following covariance matrices used to characterize the noise components of the array summations:

SQUARE-LAW ARRAY: $n(t) = n_c(t) \cos \omega t + n_s(t) \sin \omega t$

$$\mathrm{Cov}[n_c(t_1),\ n_s(t_1)] = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \tag{1-6}$$

That is, adopting the usual quadrature expansion of narrowband Gaussian processes, the sine and cosine components of the noise are considered independent at the same time instant.

---

* We shall not consider steering and beamforming as such, but mention these operations only to indicate that the models we employ can accommodate them, and to support our practice of considering only relative bearings (i.e., relative to the steering direction) within the major lobe of the array pattern.

MULTIPLICATIVE ARRAY: $n_i(t) = n_{ic}(t) \cos \omega t + n_{is}(t) \sin \omega t \quad i = 1,2$

$$\mathrm{Cov}[n_{1c}(t_1), n_{1s}(t_1), n_{2c}(t_1), n_{2s}(t_1)] =$$

$$\begin{bmatrix} \sigma_1^2 & 0 & \rho\sigma_1\sigma_2 & r\sigma_1\sigma_2 \\ 0 & \sigma_1^2 & -r\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & -r\sigma_1\sigma_2 & \sigma_2^2 & 0 \\ r\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 & 0 & \sigma_2^2 \end{bmatrix} \triangleq K_x \qquad (1\text{-}7)$$

That is, the noise correlation at the array sums is considered to be such that

$$E(n_{1c}n_{2c}) = E(n_{1s}n_{2s}) = \rho\sigma_1\sigma_2$$

$$E(n_{1c}n_{2s}) = -E(n_{1s}n_{2c}) = r\sigma_1\sigma_2. \qquad (1\text{-}8)$$

CHAPTER TWO

ARRAY PROBABILITY DENSITY FUNCTIONS

The most basic representation of the multiplicative array model we are considering is that of the product of two nonzero-mean Gaussian random variables. This description encompasses a large number of cases which are of interest, which cases are often most readily approached individually because one or more simplifying assumptions are traditionally made.

In this chapter we develop the probability density function for the generalized narrowband case of the multiplicative array configuration, showing also the related function for the square-law array. These, in turn, are specialized to the cases which are anticipated to be of most interest and, moreover, are relatively amenable to the analysis in the succeeding chapters.

The situation in which the noise is symmetric across the array and has a narrowband spectrum which is even about the center frequency is selected as the "main case." The probability density functions for the multiplicative and square-law array filter outputs are applied to this case, yielding forms in which all the array parameters appear explicitly.

Computed results are shown for the probability density functions, displaying the effects of the various distribution parameters.

The mean and variance of the filter output are calculated, and used to express the signal to noise ratio (SNR) at the output of the array processors.

2.1 PROBABILITY DENSITY FUNCTION FOR THE GENERALIZED NARROWBAND MULTIPLICATIVE ARRAY PROCESSOR

We now give our attention to a rather general case in which the multiplicative array summations of Figure 1-2 are modeled as two narrowband processes, $u_1(t)$ and $u_2(t)$, whose product $z(t)$ is passed through a zonal lowpass filter to yield the output $y(t)$, as illustrated in Figure 2-1. The processes $u_1(t)$ and $u_2(t)$ are said to consist of the super-position of known signals $s_1(t)$ and $s_2(t)$ and noise processes $n_1(t)$ and $n_2(t)$, respectively. At the same instant of time, $n_1(t)$ and $n_2(t)$ are assumed to be jointly normal random variables with variances σ_1^2 and σ_2^2, correlation coefficient ρ, and zero means, as in Section 1.2.3.

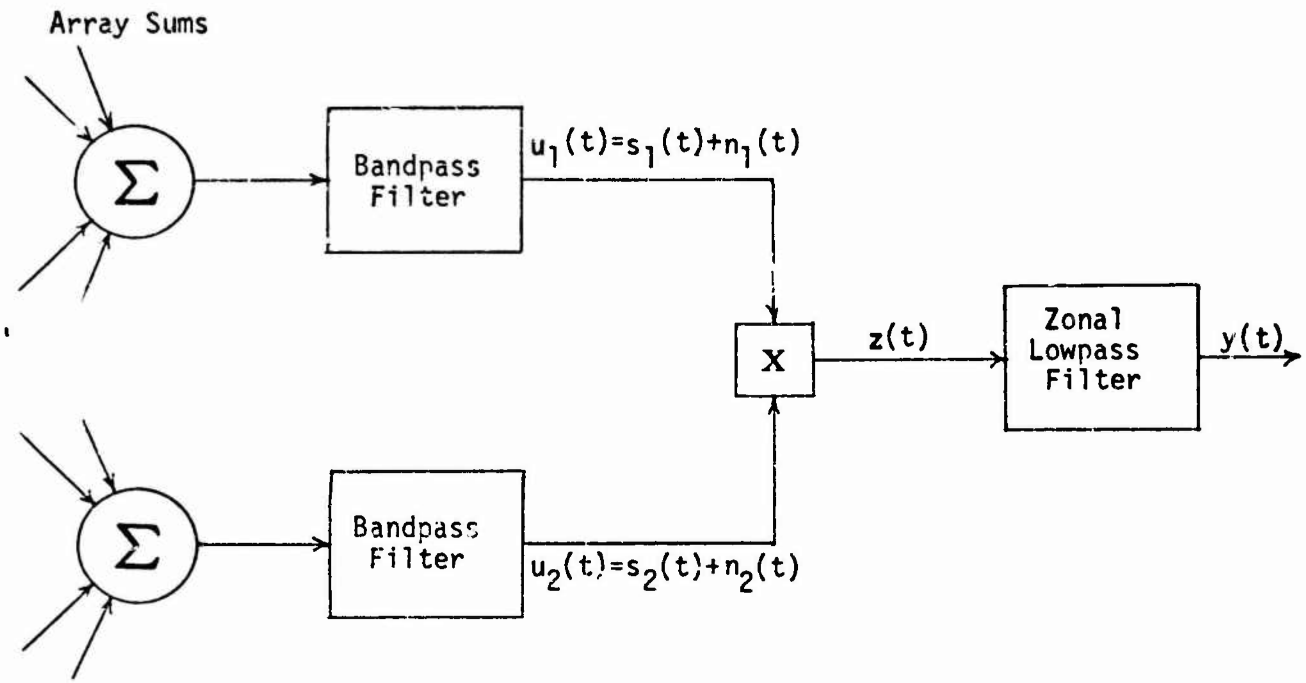
The multiplier is assumed to be instantaneous, so that its output can be written

$$\begin{aligned} z(t) &= u_1(t) \times u_2(t) \\ &= [s_1(t) + n_1(t)] \times [s_2(t) + n_2(t)] \\ &= \frac{1}{4}[u_1(t) + u_2(t)]^2 - \frac{1}{4}[u_1(t) - u_2(t)]^2 \\ &\equiv [s_3(t) + n_3(t)]^2 - [s_4(t) + n_4(t)]^2, \end{aligned} \tag{2-1}$$

where $$s_{3,4}(t) = \frac{1}{2}[s_1(t) \pm s_2(t)] \tag{2-1a}$$

and $$n_{3,4}(t) = \frac{1}{2}[n_1(t) \pm n_2(t)]. \tag{2-1b}$$

This arrangement may be recognized as the old "quarter-square multiplier" idea, used in analog computation.



$$\left.\begin{aligned} s_i(t) &= S_i(t)\cos[wt-\theta_i(t)] \\ n_i(t) &= N_i(t)\cos[wt-\Phi_i(t)] \\ &= n_{ci}(t)\cos wt+n_{si}(t)\sin wt \end{aligned}\right\} \quad i=1,\ 2$$

FIGURE 2-1 MATHEMATICAL MODEL OF MULTIPLICATIVE ARRAY

For the new variables we have defined, we have the following moments:

$$E(n_3) = 0$$

$$E(n_4) = 0 \qquad \text{(2-2a)}$$

$$E(n_3^2) \triangleq \sigma_3^2 = \frac{1}{4}(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)$$

$$E(n_4^2) \triangleq \sigma_4^2 = \frac{1}{4}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)$$

$$E(n_3 n_4) \triangleq R\sigma_3\sigma_4 = \frac{1}{4}(\sigma_1^2 - \sigma_2^2). \qquad \text{(2-2b)}$$

Note that if $\sigma_1 = \sigma_2$, then $R = 0$; that is, n_3 and n_4 are uncorrelated.

Specialization to what we are calling the narrowband case consists in the conventional assumption that the following quadrature representations apply:

$$s_3(t) = S_3(t)\cos[\omega t - \theta_3(t)], \qquad s_4(t) = S_4(t)\cos[\omega t - \theta_4(t)]$$

$$n_3(t) = n_{3c}(t)\cos\omega t + n_{3s}(t)\sin\omega t = N_3(t)\cos[\omega t - \phi_3(t)]$$

$$n_4(t) = n_{4c}(t)\cos\omega t + n_{4s}(t)\sin\omega t = N_4(t)\cos[\omega t - \phi_4(t)], \qquad \text{(2-3)}$$

where if $s_i(t) = S_i(t)\cos[\omega t - \theta_i(t)]$

$$i = 1,2 \qquad \text{(2-3a)}$$

$$n_i(t) = N_i(t)\cos[\omega t - \phi_i(t)]$$

then

$$S_{3,4}^2 = \frac{1}{4}[S_1^2 + S_2^2 \pm 2S_1S_2\cos(\theta_1 - \theta_2)]$$

$$\tan\theta_{3,4} = \frac{S_1\sin\theta_1 \pm S_2\sin\theta_2}{S_1\cos\theta_1 \pm S_2\cos\theta_2}$$

$$N_{3,4}^2 = \frac{1}{4}[N_1^2 + N_2^2 \pm 2N_1N_2\cos(\phi_1 - \phi_2)]$$

$$\tan\phi_{3,4} = \frac{N_1\sin\phi_1 \pm N_2\sin\phi_2}{N_1\cos\phi_1 \pm N_2\cos\phi_2} \qquad \text{(2-3b)}$$

With this narrowband representation, we have for the output of the multiplier,

$$z = (s_3 + n_3)^2 - (s_4 + n_4)^2$$

$$= \frac{1}{2} [S_3^2 + 2S_3N_3 \cos(\phi_3 - \theta_3) + N_3^2 - S_4^2 - 2S_4N_4 \cos(\phi_4 - \theta_4) - N_4^2]$$

$$+ \text{(terms with frequency } 2\omega). \qquad (2\text{-}4)$$

What is meant by "zonal lowpass filter" is that the filter output y is equal to the multiplier output z, less the terms of frequency 2ω. Thus if we define the sum and difference terms

$$z_1(t) \triangleq s_3(t) + n_3(t) = \frac{1}{2} [u_1(t) + u_2(t)] = Z_1(t) \cos[\omega t - \phi_1(t)]$$

$$= z_{1c}(t) \cos \omega t + z_{1s}(t) \sin \omega t \qquad (2\text{-}4a)$$

$$z_2(t) \triangleq s_4(t) + n_4(t) = \frac{1}{2} [u_1(t) - u_2(t)] = Z_2(t) \cos[\omega t - \phi_2(t)]$$

$$= z_{2c}(t) \cos \omega t + z_{2s}(t) \sin \omega t \qquad (2\text{-}4b)$$

we find that

$$y(t) = \frac{1}{2} [Z_1^2(t) - Z_2^2(t)] \qquad (2\text{-}5)$$

where

$$Z_{1,2}^2 = \frac{1}{4} [S_1^2 + S_2^2 + N_1^2 + N_2^2 \pm 2S_1S_2 \cos(\theta_2 - \theta_1)$$

$$+ 2S_1N_1 \cos(\phi_1 - \theta_1) \pm 2S_1N_2 \cos(\phi_2 - \theta_1)$$

$$\pm 2S_2N_1 \cos(\phi_1 - \theta_2) + 2S_2N_2 \cos(\phi_2 - \theta_2) \pm 2N_1N_2 \cos(\phi_1 - \phi_2)]$$

$$= S_{3,4}^2 + N_{3,4}^2 + 2S_{3,4}N_{3,4} \cos(\phi_{3,4} - \theta_{3,4}) \qquad (2\text{-}5a)$$

and

$$\tan\phi_{1,2} = \frac{S_1 \sin\theta_1 \pm S_2 \sin\theta_2 + N_1 \sin\phi_1 \pm N_2 \sin\phi_2}{S_1 \cos\theta_1 \pm S_2 \cos\theta_2 + N_1 \cos\phi_1 \pm N_2 \cos\phi_2}$$

$$= \frac{S_{3,4} \sin\theta_{3,4} + N_{3,4} \sin\phi_{3,4}}{S_{3,4} \cos\theta_{3,4} + N_{3,4} \cos\phi_{3,4}} \qquad (2\text{-}5b)$$

Our object is to find the probability density function of the filter output y(t) at a given instant in time. (From this point on, reference to time will be suppressed.)

2.1.1 DISTRIBUTION OF THE SUM AND DIFFERENCE TERMS

If, to use vector notation, we refer to the input noise components as $\underline{x}' = (n_{1c}, n_{1s}, n_{2c}, n_{2s})$, where the prime (') is used to indicate the transpose, then we may express the joint probability density function of the input noise components as

$$p_0(\underline{x}) = \frac{1}{4\pi^2\sqrt{\det K_x}} \exp[-\frac{1}{2}\underline{x}'K_x^{-1}\underline{x}],$$

where K_x, the covariance matrix, is postulated to be

$$K_x = \mathrm{Cov}[n_{1c}(t), n_{1s}(t), n_{2c}(t), n_{2s}(t)]$$

$$= E[\underline{x}'\underline{x}]$$

$$\equiv \begin{bmatrix} \sigma_1^2 & 0 & \rho\sigma_1\sigma_2 & r\sigma_1\sigma_2 \\ 0 & \sigma_1^2 & -r\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & -r\sigma_1\sigma_2 & \sigma_2^2 & 0 \\ r\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 & 0 & \sigma_2^2 \end{bmatrix}$$

That is, the noise correlation at the inputs to the multiplier is considered to be such that, for the same time instant,

$$E(n_{1c}n_{2c}) = E(n_{1s}n_{2s}) = \rho\sigma_1\sigma_2$$

$$E(n_{1c}n_{2s}) = -E(n_{1s}n_{2c}) = r\sigma_1\sigma_2.$$

We may consider the sum and difference noise components $\underline{n}' = (n_{3c}, n_{3s}, n_{4c}, n_{4s})$ as a linear transofrmation $\underline{n} = A\underline{x}$, where the matrix A is given by

$$A = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \end{bmatrix}.$$

We may then write the probability density function of the variables $\underline{n}$ as

$$p_1(\underline{n}) = \frac{1}{4\pi^2 D} \exp[-\frac{1}{2}\underline{n}'K_n^{-1}\underline{n}], \tag{2-6}$$

where the new covariance matrix K_n is easily computed to be

$$K_n = AK_xA = \begin{bmatrix} \sigma_3^2 & 0 & R\sigma_3\sigma_4 & R'\sigma_3\sigma_4 \\ 0 & \sigma_3^2 & -R'\sigma_3\sigma_4 & R\sigma_3\sigma_4 \\ R\sigma_3\sigma_4 & -R'\sigma_3\sigma_4 & \sigma_4^2 & 0 \\ -R'\sigma_3\sigma_4 & R\sigma_3\sigma_4 & 0 & \sigma_4^2 \end{bmatrix} \tag{2-7}$$

and $D = \sqrt{\det K_n} = \sigma_3^2\sigma_4^2[1 - R^2 - (R')^2]$. (2-7a)

The variances σ_3^2 and σ_4^2, and R are as given in (2-2), and $R'\sigma_3\sigma_4 = -\frac{1}{2} r\sigma_1\sigma_2$.

It follows then that the quadrature expansions of the signal plus noise variables z_1 and z_2 as defined above have the density function

$$p_2(z_{1c},z_{1s},z_{2c},z_{2s}) = \frac{1}{4\pi^2 D} \exp[-\frac{1}{2}(\underline{z}-\underline{s})'K_n^{-1}(\underline{z}-\underline{s})] \tag{2-8}$$

where $(\underline{z}-\underline{s})' = (z_{1c}-S_3 \cos\theta_3, z_{1s}-S_3 \sin\theta_3, z_{2c}-S_4 \cos\theta_4, z_{2s}-S_4 \sin\theta_4)$ (2-9)

2.1.2 DISTRIBUTION OF THE FILTER OUTPUT, y

Recall that the filter output is $y = \frac{1}{2}(Z_1^2 - Z_2^2)$. The density function of Z_1 and Z_2 is

$$p_4(Z_1,Z_2) = \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2\, p_3(Z_1,\phi_1,Z_2,\phi_2)$$

$$= Z_1Z_2 \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2\, p_2(Z_1 \cos\phi_1, Z_1 \sin\phi_1, Z_2 \cos\phi_2, Z_2 \sin\phi_2)$$

(2-10a)

$$= \frac{Z_1 Z_2}{4\pi^2 D} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \exp[-\tfrac{1}{2}(\underline{Z}-\underline{s})' K_n^{-1}(\underline{Z}-\underline{s})] \qquad (2\text{-}10b)$$

where $(\underline{Z} - \underline{s})' = (Z_1 \cos\phi_1 - S_3 \cos\theta_3, Z_1 \sin\phi_1 - S_3 \sin\theta_3,$

$$Z_2 \cos\phi_2 - S_4 \cos\theta_4, Z_2 \sin\phi_2 - S_4 \sin\theta_4). \qquad (2\text{-}11)$$

The quadratic form appearing in the integral's exponent reduces to

$$\begin{aligned} Q = \frac{-1}{2D} [&\sigma_4^2(Z_1^2 + S_3^2) + \sigma_3^2(Z_2^2 + S_4^2) - 2\sigma_3\sigma_4 X S_3 S_4 \cos(\theta_4-\theta_3-x) \\ &-2\sigma_3\sigma_4 X Z_1 Z_2 \cos(\phi_1-\phi_2-x) \\ &-2\sigma_4 V Z_1 \cos(\phi_1-v) - 2\sigma_3 W Z_2 \cos(\phi_2-w)] \end{aligned} \qquad (2\text{-}12)$$

where $X^2 = R^2 + (R')^2$, $\tan x = R'/R$

$$V^2 = \sigma_4^2 S_3^2 + \sigma_3^2 X^2 S_4^2 - 2\sigma_3\sigma_4 X S_3 S_4 \cos(\theta_3-\theta_4+x)$$

$$W^2 = \sigma_3^2 S_4^2 + \sigma_4^2 X^2 S_3^2 - 2\sigma_3\sigma_4 X S_3 S_4 \cos(\theta_3-\theta_4+x)$$

$$\tan v = \frac{\sigma_4 S_3 \sin\theta_3 - \sigma_3 X S_4 \sin(\theta_4-x)}{\sigma_4 S_3 \cos\theta_3 - \sigma_3 X S_4 \cos(\theta_4-x)}$$

$$\tan w = \frac{\sigma_3 S_4 \sin\theta_4 - \sigma_4 X S_3 \sin(\theta_3+x)}{\sigma_3 S_4 \cos\theta_4 - \sigma_4 X S_3 \cos(\theta_3+x)} \qquad (2\text{-}13)$$

Using the relationship $e^{x\cos w} = \sum_{n=0}^{\infty} \varepsilon_n I_n(x) \cos nw$, $\varepsilon_n = \begin{cases} 1, n = 0 \\ 2, n > 0 \end{cases}$ and performing the integration with respect to ϕ_1 and ϕ_2, in a manner quite similar to Middleton[16] (chapter 9) we obtain

$$p_4(Z_1,Z_2) = \frac{Z_1Z_2}{D} \exp\left\{\frac{-1}{2D}\ [\sigma_4^2 Z_1^2 + \sigma_3^2 Z_2^2 + U^2]\right\}$$

$$\times \sum_{m=0}^{\infty} \epsilon_m I_m \left(\frac{\sigma_4 VZ_1}{D}\right) I_m \left(\frac{\sigma_3 WZ_2}{D}\right) I_m \left(\frac{\sigma_3\sigma_4 XZ_1Z_2}{D}\right) \cos m(v-w-x) \quad (2\text{-}14)$$

where we have used $U^2 = \sigma_4^2 S_3^2 + \sigma_3^2 S_4^2 - 2\sigma_3\sigma_4 XS_3S_4 \cos(\theta_4-\theta_3-x)$, (2-15)

and $I_m(\ )$ is the m:th order modified Bessel function of the first kind.

To obtain the density for y, we use the transformations

$$\begin{cases} Z_1 = \sqrt{2y}\cosh u \\ Z_2 = \sqrt{2y}\sinh u \end{cases} \quad \text{for } Z_1 > Z_2, \text{ or } y > 0 \text{ and } u > 0$$

and

$$\begin{cases} Z_1 = \sqrt{-2y}\sinh u \\ Z_2 = \sqrt{-2y}\cosh u \end{cases} \quad \text{for } Z_1 < Z_2, \text{ or } y < 0 \text{ and } u > 0. \quad (2\text{-}16)$$

Then we find that the density for y is given by the expression

$$p_6(y) = \int_0^\infty du\ p_5(y,u) = \int_0^\infty du\ p_4(\sqrt{2y}\cosh u\ ,\ \sqrt{2y}\sinh u\ ),\ y > 0$$

$$= \int_0^\infty du\ p_4(\sqrt{-2y}\sinh u\ ,\ \sqrt{-2y}\cosh u\ ),\ y < 0.$$

$$= \frac{|y|}{D} \exp\left[\frac{-(\sigma_3^2-\sigma_4^2)|y|-U^2}{2D}\right] \sum_{m=0}^{\infty} \epsilon_m \cos m(v-w-x)$$

$$\times \int_0^\infty du\ \sinh 2u\ \ e^{-a\cosh 2u}\ I_m(b\cosh u\ )I_m(c\sinh u\ )I_m(d\sinh 2u)$$

(2-17)

with $\quad a = |y|(\sigma_3^2+\sigma_4^2)/2D, \quad d = \sigma_3\sigma_4 X|y|/D$, and

$$(b,c) = (\sigma_4 V\sqrt{2y}/D, \ \sigma_3 W\sqrt{2y}/D), \ y > 0$$

$$= (\sigma_3 W\sqrt{-2y}/D, \ \sigma_4 V\sqrt{-2y}/D), \ y < 0. \tag{2-18}$$

By writing $x = 1 + \cosh 2u$, we may change the above integral to

$$\frac{1}{2} e^a \int_0^\infty dx\, e^{-ax} I_m[b\sqrt{\tfrac{1}{2}(x+2)}]\ I_m\ [c\sqrt{\tfrac{1}{2}x}]\ I_m\ [d\sqrt{x(x+2)}]$$

$$= \frac{1}{2} e^a \sum_{n=0}^\infty \sum_{k=0}^\infty \frac{(b/2\sqrt{2})^{2n+m}}{n!(n+m)!} \frac{(\frac{1}{2}d)^{2k+m}}{k!(k+m)!} \int_0^\infty dx\, e^{-ax}(x+2)^{n+k+m} x^{k+\frac{1}{2}m}$$

$$\times\ I_m[c\sqrt{\tfrac{1}{2}x}]$$

$$= \frac{1}{2} e^a \sum_{n=0}^\infty \sum_{k=0}^\infty \sum_{r=0}^{n+k+m} \frac{(b/2\sqrt{2})^{2n+m}}{n!(n+m)!} \frac{(\frac{1}{2}d)^{2k+m}}{k!(k+m)!} \binom{n+k+m}{r} 2^{n+k+m-r}$$

$$\times \int_0^\infty dx\, e^{-ax} x^{k+\frac{1}{2}m+r} I_m[c\sqrt{\tfrac{1}{2}x}] \tag{2-19}$$

Making use of Gradshteyn and Ryzhik[13] (formulas 6.643.4 and 9.220.2), we obtain for (2-19) the following expression:

$$(\frac{c}{2a})^m \frac{1}{2a} \exp(a+ \frac{c^2}{8a}) \sum_{n=0}^\infty \sum_{k=0}^\infty \sum_{r=0}^{n+k+m} \frac{(\frac{1}{2}b)^{2n+m}(\frac{1}{2}d)^{2k+m}}{n!(n+m)!k!(k+m)!} \binom{n+k+m}{r} \frac{2^{k-r}}{a^{k+r}} (k+r)!$$

$$\times\ L_{k+r}^m(\frac{-c^2}{8a}) \tag{2-20}$$

where the $L^m_{k+r}(\ )$ are the Laguerre polynomials of order k+r and parameter m. Substituting in (2-17) and for a, b, c, and d, and rearranging terms for clarity, we have finally

$$p_6(y) = \frac{1}{\sigma_3^2+\sigma_4^2} \exp\left\{\frac{-1}{2D}[2\sigma_0^2|y|+U^2-g^2/(\sigma_3^2+\sigma_4^2)]\right\}$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n+k+m}\left(\frac{fg\sigma_3\sigma_4X}{2D^2(\sigma_3^2+\sigma_4^2)}\right)^m\left(\frac{f^2}{2D^2}\right)^n\left(\frac{\sigma_3^2\sigma_4^2X^2}{D(\sigma_3^2+\sigma_4^2)}\right)^k\left(\frac{D}{\sigma_3^2+\sigma_4^2}\right)^r$$

$$\times\ \epsilon_m \cos m(v-w-x)\ \frac{(k+r)!}{n!(n+m)!k!(k+m)!}\binom{k+n+m}{r}$$

$$\times\ |y|^{k+m+n-r} L^m_{k+r}\left(\frac{-g^2}{2D(\sigma_3^2+\sigma_4^2)}\right) \tag{2-21}$$

with

$$(f,g,\sigma_0^2) = \begin{cases}(\sigma_4V,\ \sigma_3W,\ \sigma_4^2), & y > 0\\[2ex] (\sigma_3W,\ \sigma_4V,\ \sigma_3^2), & y < 0.\end{cases} \tag{2-22}$$

A recent paper by Andrews[17] gives an expression for the density function in nearly the same situation, by means of the characteristic function method. Comparing the development here with that of Andrews, it appears that following the strenuous development associated with the computations of the characteristic function, one still needs to solve a convolution integral such as our (2-19) above. The different final expression he obtains results simply from a different method of attacking this type of integral. For the special cases, of course, the form of the expressions given here is identical to that of Andrews. Moreover, the computed results we show below match his wherever applicable.

2.1.3 REDUCTION OF THE GENERAL EXPRESSION TO SPECIAL CASES

For practical applications and for purposes of checking our results with those of previous authors, it is instructive to reduce the general expression of (2-21) under certain assumptions.

2.1.3.1 Equal Noise Power with Even Spectrum

An important case is that for which the noise inputs are of equal power and their spectra are even about the center frequency of the band. Under these assumptions, the cross-quadrature correlation coefficient r is zero[16] and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, implying that $R = R' = 0$, so that the term X in (2-21) is zero. Defining the input SNR for channels 1 and 2 by

$$h_1^2 \triangleq \frac{S_1^2}{2\sigma^2}$$

and

$$h_2^2 \triangleq \frac{S_2^2}{2\sigma^2}$$

the general expression of (2-21) is reduced to the following form:

For $y > 0$,

$$p_6(y) = \frac{1}{\sigma^2} \exp\left\{-\frac{2y}{\sigma^2(1+\rho)} - h_3^2 - \left(\frac{1-\rho}{2}\right) h_4^2\right\}$$

$$\times \sum_{r=0}^{\infty} \left[\frac{h_3(1-\rho)\sqrt{1+\rho}}{\sqrt{8y/\sigma^2}}\right]^r I_r\left(\frac{2h_3}{\sigma}\sqrt{\frac{2y}{1+\rho}}\right) L_r\left[-\left(\frac{1+\rho}{2}\right) h_4^2\right] \quad (2\text{-}23a)$$

and for $y < 0$

$$p_6(y) = \frac{1}{\sigma^2} \exp\left\{\frac{2y}{\sigma^2(1-\rho)} - \left(\frac{1+\rho}{2}\right) h_3^2 - h_4^2\right\}$$

$$\times \sum_{r=0}^{\infty} \left[\frac{h_4(1+\rho)\sqrt{1-\rho}}{\sqrt{-8y/\sigma^2}}\right]^r I_r\left(\frac{2h_4}{\sigma}\sqrt{\frac{-2y}{1-\rho}}\right) L_r\left[-\frac{(1-\rho)}{2} h_3^2\right] \tag{2-23b}$$

where

$$h_3^2 = \frac{1}{2(1+\rho)}\left(h_1^2+h_2^2+2h_1h_2 \cos 2\theta\right) \tag{2-23c}$$

$$h_4^2 = \frac{1}{2(1-\rho)}\left(h_1^2+h_2^2-2h_1h_2 \cos 2\theta\right) \tag{2-23d}$$

and where $2\theta = \theta_1 - \theta_2$ is the difference in phase between the narrowband signals at the multiplier input [see Figure 2-1 and (2-3a)]. Muraka[18] presents results corresponding to a further specialization of this case to $\theta = 0$ and $h_1^2 = h_2^2$.

2.1.3.2 No Input Signals and Equal Noise Power

If the correlator inputs are assumed to be only the noise of equal power, that is, $h_1^2 = h_2^2 = 0$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the pdf (2-21) is further reduced to

$$p_6(y) = \begin{cases} \frac{1}{\sigma^2} \exp\left[-\frac{2y}{\sigma^2(1+\rho)}\right], & y > 0 \\ \\ \frac{1}{\sigma^2} \exp\left[\frac{2y}{\sigma^2(1-\rho)}\right], & y < 0 \end{cases} \tag{2-24}$$

This result is identical to that of Lezin[19], who began analysis with these assumptions.

2.1.4 THE PROBABILITY DENSITY FUNCTION FOR THE OUTPUT OF THE STANDARD OR SQUARE LAW ARRAY DETECTOR

In the ensuing chapters we shall have use for the density function for the standard or square law array scheme, as we have defined it in Section 1.2.2. If the input to the squaring device is $z(t) = s(t) + n(t) = Z(t) \cos [\omega t-\phi(t)]$, then the filter output is given by $y = Z^2/2$.

As above, we may write $Z^2 = z_c^2 + z_s^2$, where z_c and z_s are independent Gaussian variates with identical variance σ^2 and means $S\cos\theta$ and $S\sin\theta$ respectively if $s(t) = S(t) \cos [\omega t-\theta(t)]$.

It is well known that the sum of the squares of n independent, unit-variance Gaussian variates with means m_k is a noncentral chi-square variate with n degrees of freedom and noncentrality parameter $m_1^2 + m_2^2 + \ldots + m_n^2$ [23]. Therefore, $x = \frac{2y}{\sigma^2}$ is noncentral chi-square with two degrees of freedom and parameter S^2/σ^2, and the probability density function of y is

$$P_{sq}(y) = \frac{2}{\sigma^2} \chi^2\left[\frac{2y}{\sigma^2}; 2, \frac{S^2}{\sigma^2}\right] = \frac{1}{\sigma^2} \exp\left(-h^2 - \frac{y}{\sigma^2}\right) I_0\left(2h\sqrt{y/\sigma^2}\right), \; y > 0, \quad (2\text{-}25)$$

where h^2 is the SNR at the input to the square law device.

2.1.5 COMPUTED RESULTS

Judging the symmetric array, even spectrum case to be one of most general utility, we shall present computations of this probability density function (2-23), as well as the related density for the square law array (2-25), in order to show the effects of the parameters ρ, $h_1 = h_2 = h^2$, and $\theta = (\theta_1 - \theta_2)/2$; σ, quite obviously, is a scaling parameter. In Appendix B the pdf (2-23) is shown to be equivalent to

$$p_6(y) = \frac{1}{\sigma^2} \exp\left[-(h_3^2 + h_4^2)\right] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{[\frac{1}{2}(1-\rho)h_3^2]^m}{m!} \frac{[\frac{1}{2}(1+\rho)h_4^2]^n}{n!}$$

$$\times \begin{cases} \exp\left[\dfrac{-2y}{\sigma^2(1+\rho)}\right] G_m^n \left[\dfrac{4y}{\sigma^2(1-\rho^2)}\right], & y \geq 0 \\ \exp\left[\dfrac{2y}{\sigma^2(1-\rho)}\right] G_n^m \left[\dfrac{-4y}{\sigma^2(1-\rho^2)}\right], & y < 0, \end{cases} \qquad (2\text{-}26)$$

where the polynomials we have defined, $G_m(x) = \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k}{k!}$ have the unusually useful computational property $G_m^n = G_{m-1}^n + G_m^{n-1}$. The form (2-26) results from application of the characteristic function method to the special case we are now considering.

For computation of (2-26), we have chosen a "nominal case" of specific values for the parameters: $\rho=0$, $\sigma=1$, $h=1$, $\theta=0$. In Figures 2-2 through 2-5, (2-26) and (2-25) are plotted under varying parameters. It is seen in Figure 2-2, that h^2 has the powerful effect of changing the pointed curve of the no signal case into curves which begin to approach the familiar Guassian.

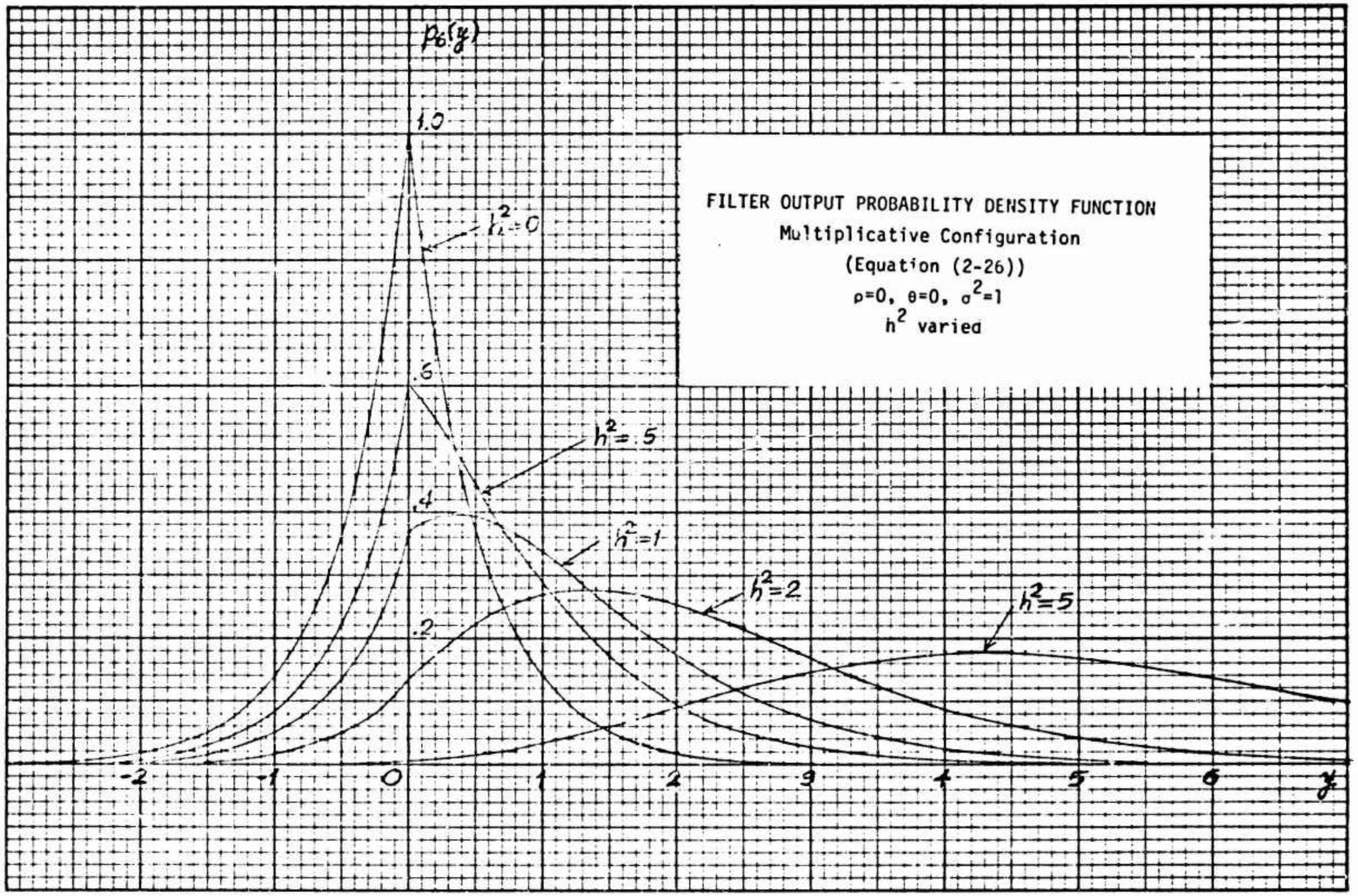


FIGURE 2-2 MULTIPLICATIVE CONFIGURATION PDF; SNR VARIED

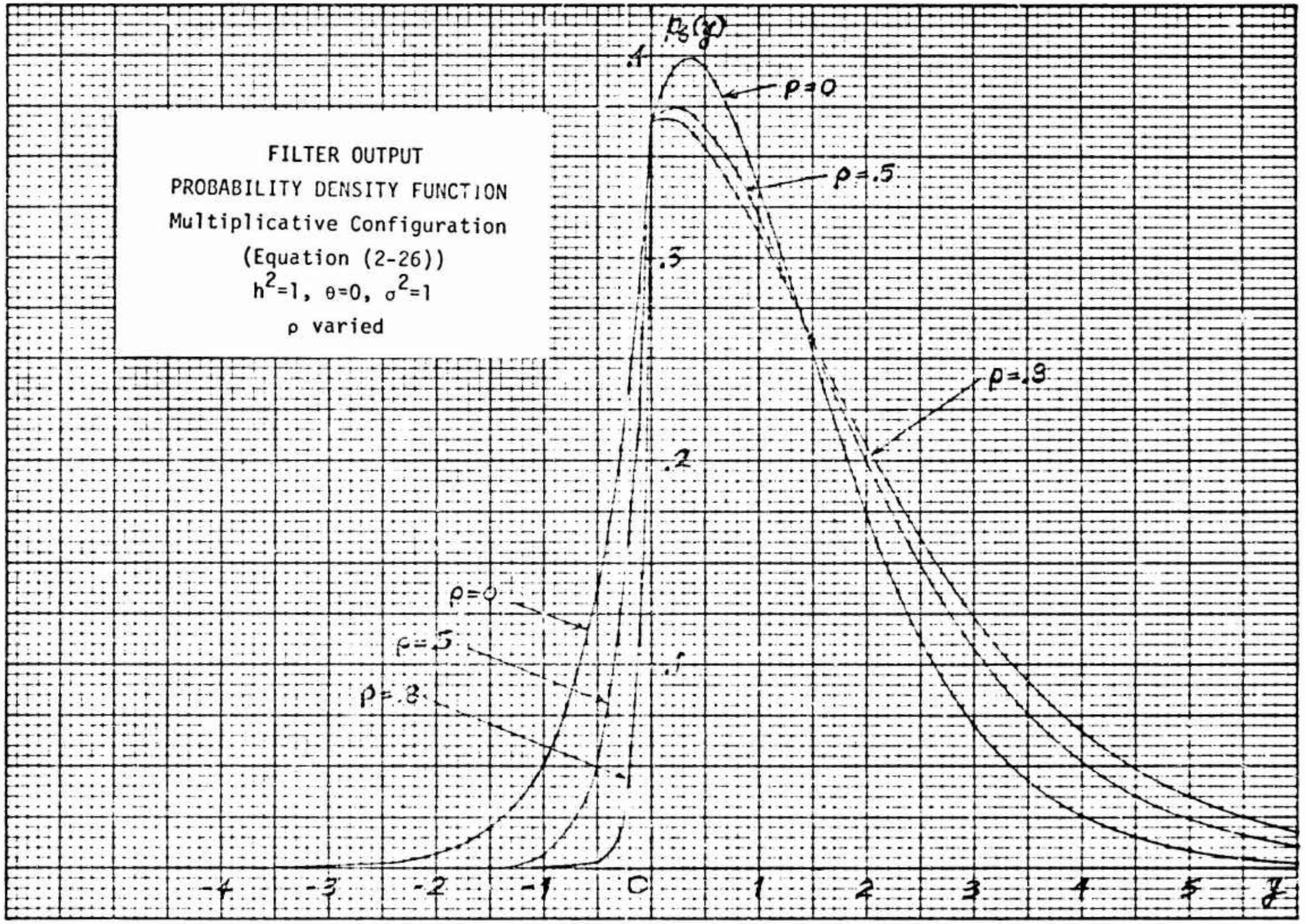


FIGURE 2-3 MULTIPLICATIVE CONFIGURATION PDF; CHANNEL CORRELATION VARIED

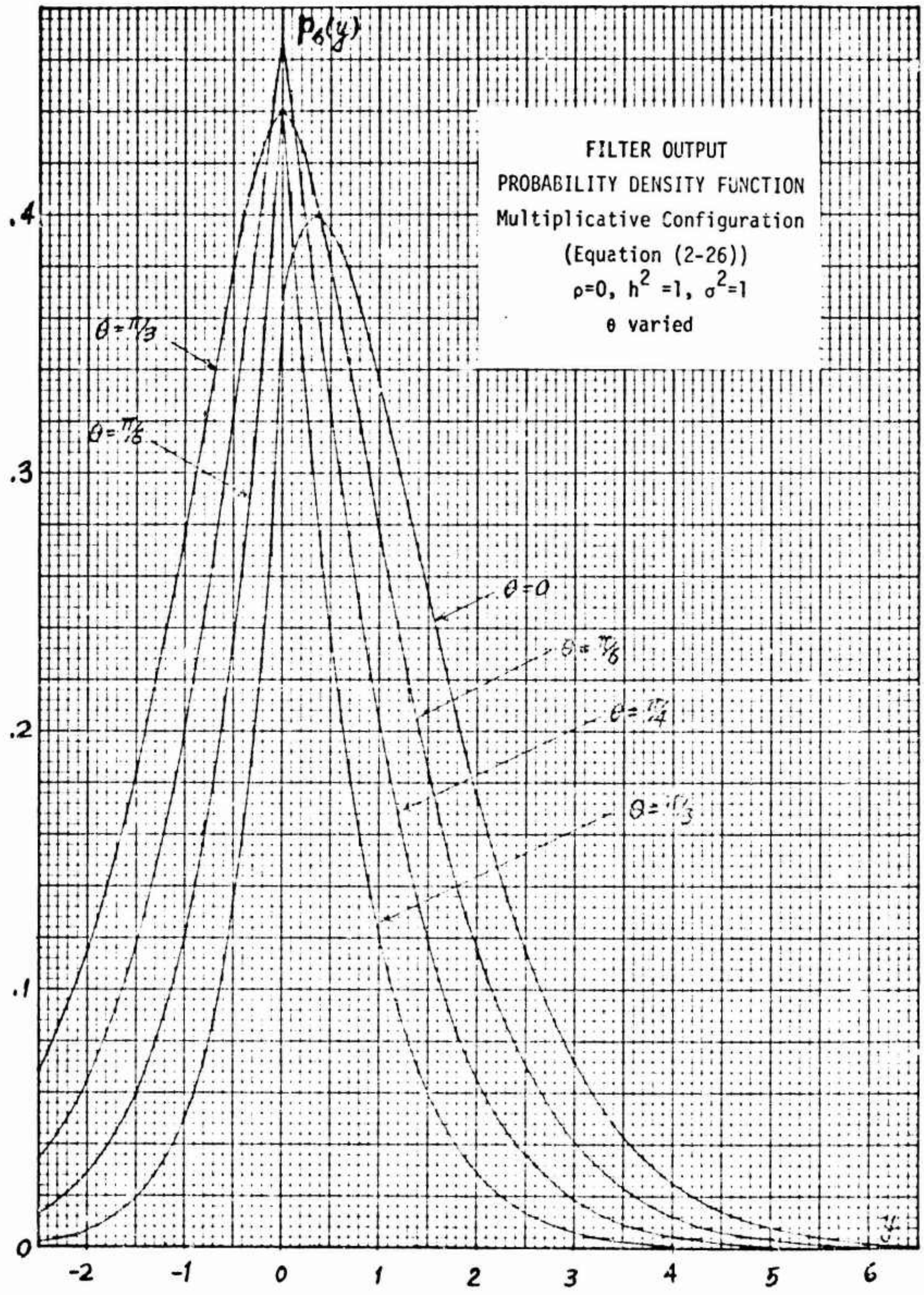


FIGURE 2-4 MULTIPLICATIVE CONFIGURATION PDF; PHASE VARIED

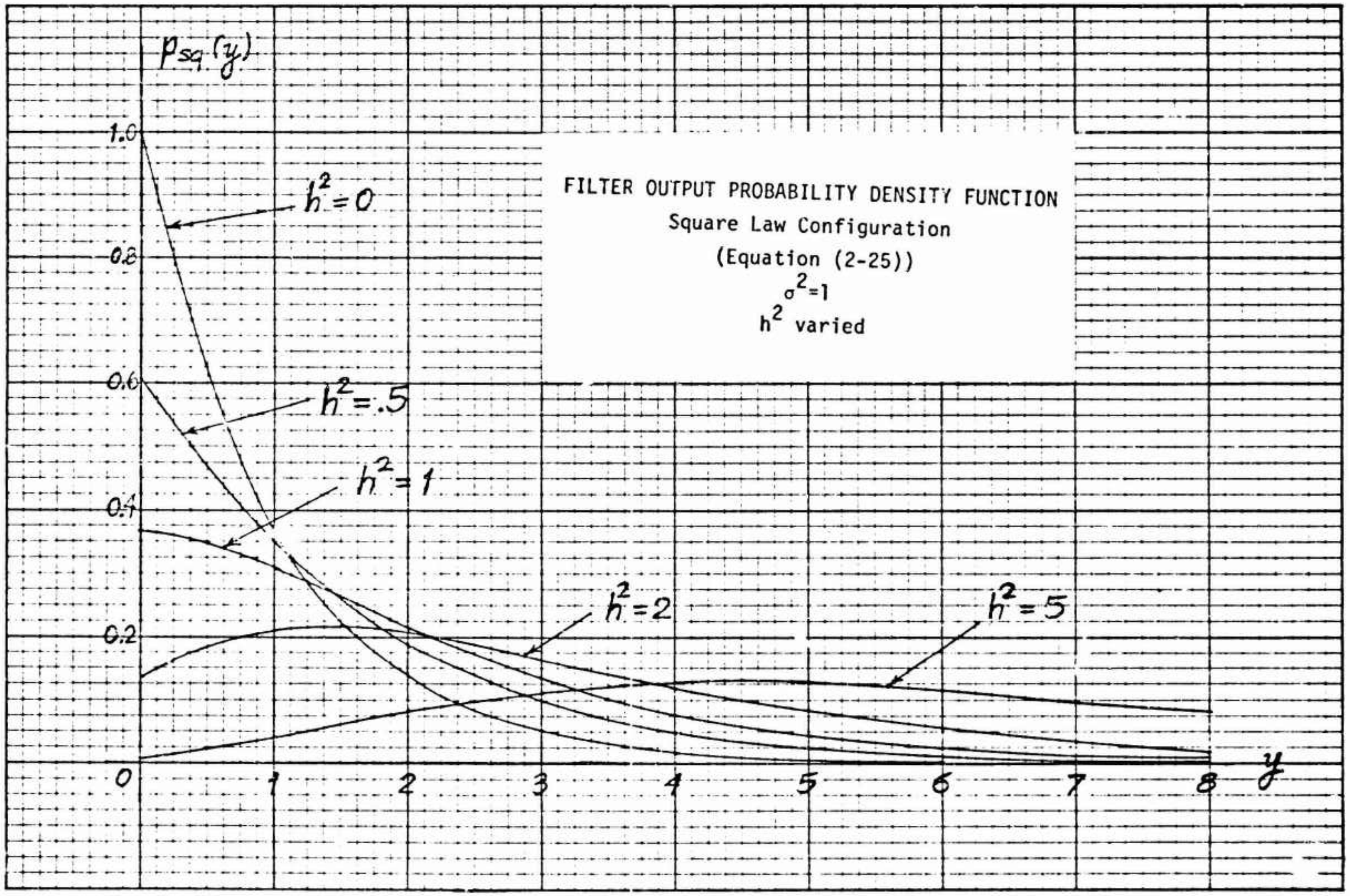


FIGURE 2-5 STANDARD CONFIGURATION PDF; SNR VARIED

2.2 APPLICATION TO THE ARRAY MODELS

All subsequent analysis shall be based on the application of the filter output probability density functions for the "main" case in which the array and the array noise input processes are symmetric, and the noise spectrum is even about the narrowband center frequency. This application shall consist, in this chapter, in showing the explicit probability density functions (that is, containing all the pertinent parameters) for either type of array, including computed results.

In order to be able to show the effects of inter-element correlation on the probability density, we have taken pains in Appendix A, Section 3, to obtain closed form representations for the variances and the correlation coefficient at the array sums. The resulting complexity in expression is considerable, but we shall be quick to illustrate with special cases, so that the train of thought is not lost.

2.2.1 THE EXPLICIT PROBABILITY DENSITY FUNCTION FOR THE MULTIPLICATIVE ARRAY MODEL

For the multiplicative array model, from (1-5) it is evident that we have for the array sum signal parameters,

$$S_1 = S_2 = MAD(\theta/M;\ M),\ \tfrac{1}{2}(\theta_1 - \theta_2) = \theta = \tfrac{1}{2}M\beta\sin\alpha, \tag{2-27}$$

where A is the signal input amplitude, the total number of array elements is 2M, α is the relative bearing, $D(x;n) = \frac{\sin NX}{N\sin X}$, $\beta = \frac{2\pi d}{\lambda}$.

Referring now to the two models for inter-element correlation in Appendix A, we have for the multiplicative array,

$$\sigma^2 = Ms^2 + 2(M-1)as^2$$

ADJACENT CORRELATION

$$\rho = \frac{a}{M + 2(M-1)a}$$ a = correlation coefficient (2-28)

$$h^2 = \frac{M^2H^2D^2(\theta/M;\ M)}{M + 2(M-1)a}$$

$$\sigma^2 = Ms^2 + 2M(M-1)s^2\left(\frac{b+e^{-b}-1}{b^2}\right)$$

$$\rho = \frac{Me^{-b}(1-e^{-b})^2}{b^2+2(M-1)(b+e^{-b}-1)}$$

EXPONENTIALLY DECREASING CORRELATION (2-29)

Correlation between elements m and n $\propto e^{-b|m-n|}$

$$h^2 = \frac{b^2MH^2D^2(\theta/M;\ M)}{b^2+2(M-1)(b+e^{-b}-1)}$$

where $s^2 \triangleq$ noise variance at hydrophones

$H^2 \triangleq$ SNR at hydrophones.

In order to keep the notation under control, let us define for either model of correlation the two parameters

$$B = \sigma^2/Ms^2$$

$$C = \rho B. \qquad (2\text{-}30)$$

Thus we may conveniently resort to the white noise case by setting B = 1 and C = 0. With these definitions, we may substitute into (2-23) to obtain the explicit probability density function for the multiplicative array filter output:

$$p_6(y) = p_6(y;\ \alpha,\ \beta,\ M,\ s,\ H,\ a \text{ or } b)$$

$$= \frac{1}{BMs^2} \exp\left[\frac{-2y}{Ms^2(B+C)} - \frac{2MH^2D^2}{B}\,\frac{B-C\cos 2\theta}{B^2-C^2} + \frac{MH^2D^2\sin^2\theta}{B}\,\frac{B+C}{B-C}\right]$$

$$\times \sum_{r=0}^{\infty}\left(\frac{MsHD\cos\theta}{2\sqrt{y}}\,\frac{B-C}{B}\right)^r I_r\left(\frac{4HD\cos\theta\sqrt{y}}{s(B+C)}\right) L_r\left(\frac{-MH^2D^2\sin^2\theta}{B}\,\frac{B+C}{B-C}\right),\quad y > 0 \tag{2-31}$$

$$= \frac{1}{BMs^2} \exp\left[\frac{2y}{Ms^2(B-C)} - \frac{2MH^2D^2}{B}\,\frac{B-C\cos 2\theta}{B^2-C^2} + \frac{MH^2D^2\cos^2\theta}{B}\,\frac{B-C}{B+C}\right]$$

$$\times \sum_{r=0}^{\infty}\left(\frac{MsHD\sin\theta}{2\sqrt{-y}}\,\frac{B+C}{B}\right)^r I_r\left(\frac{4HD\sin\theta\sqrt{-y}}{s(B-C)}\right) L_r\left(\frac{-MH^2D^2\cos^2\theta}{B}\,\frac{B-C}{B+C}\right),\quad y < 0. \tag{2-32}$$

To illustrate the use of this expression, we shall give four examples. To obtain the white noise case, we use $B = 1$ and $C = 0$, with the result

$$p_6(y) = p_6(y;\ \alpha,\ \beta,\ M,\ s,\ H)$$

$$= \frac{1}{Ms^2} \exp\left(-2y/Ms^2 - 2MH^2D^2 + MH^2D^2\sin^2\theta\right)$$

$$\times \sum_{r=0}^{\infty}\left(\frac{MsHD\cos\theta}{2\sqrt{y}}\right)^r I_r\left(\frac{4HD\cos\theta\sqrt{y}}{s}\right) L_r(-MH^2D^2\sin^2\theta),\quad y > 0 \tag{2-33}$$

$$= \frac{1}{Ms^2} \exp\left(2y/Ms^2 - 2MH^2D^2 + MH^2D^2\cos^2\theta\right)$$

$$\times \sum_{r=0}^{\infty}\left(\frac{MsHD\sin\theta}{2\sqrt{-y}}\right)^r I_r\left(\frac{4HD\sin\theta\sqrt{-y}}{s}\right) L_r(-MH^2D^2\cos^2\theta),\quad y < 0. \tag{2-34}$$

For a second example, let us consider a 10-element multiplicative array with adjacent correlation. Then $M = 5$, and for $y > 0$,

$$
\begin{aligned}
& p_6(y;\ \alpha,\ \beta,\ 5,\ s,\ H,\ a) \\
& \quad = \frac{1}{5s^2(1+1.6a)} \exp\left\{ \frac{-2y}{5s^2(1+2.6a)} - \frac{10H^2D^2}{1+1.6a}\ \frac{1+1.6a-a\cos 2\theta}{1+3.2a+1.56a^2} \right\} \\
& \quad \times \exp\left\{ \frac{5H^2D^2\sin\theta}{1+1.6a}\ \frac{1+2.6a}{1+.6a} \right\} \sum_{r=0}^{\infty} \left(\frac{2.5sHD\cos\theta}{\sqrt{y}}\ \frac{1+.6a}{1+1.6a} \right)^r \\
& \quad \times I_r\left(\frac{4HD\cos\theta\sqrt{y}}{s(1+1.6a)} \right) L_r\left(\frac{-5H^2D^2\sin^2\theta}{1+1.6a}\ \frac{1+2.6a}{1+.6a} \right) \qquad \text{(2-35)}
\end{aligned}
$$

For our third example, let us find the density for an array whose noise environment has been determined to be exponentially decreasing, in the manner in which we have defined this concept, and for which experiments yield the information $b = 1$. Thus $B = .74$ and $C = .15$. We have for this density, if $y > 0$,

$$
\begin{aligned}
& p_6(y;\ \alpha,\ \beta,\ M,\ s,\ H,\ b=1) \\
& \quad = \frac{1}{Ms^2(.26+.74M)} \exp\left\{ \frac{-2y}{Ms^2(.41+.74M)} - \frac{2MH^2D^2}{.26+.74M}\ \frac{.26+.74M-.15\cos 2\theta}{.045+.34M+.55M^2} \right\} \\
& \quad \times \exp\left\{ \frac{MH^2D^2\sin^2\theta}{.26+.74M}\ \frac{.41+.74M}{.11+.74M} \right\} \sum_{r=0}^{\infty} \left(\frac{MsHD\cos\theta}{2\sqrt{y}}\ \frac{.11+.74M}{.26+.74M} \right)^r \\
& \quad I_r\left[\frac{4HD\cos\theta\sqrt{y}}{s(.41+.74M)} \right] L_r\left(\frac{-MH^2D^2\sin^2\theta}{.26+.74M}\ \frac{.41+.74M}{.11+.74M} \right). \qquad \text{(2-36)}
\end{aligned}
$$

As a fourth example, let us find what is called the "boresight" case, where $\alpha = 0$. Then $\theta = 0$ and $D = 1$, so that

$$p_6(y;\ 0,\ \beta,\ M,\ s,\ H,\ a \text{ or } b)$$

$$= \frac{1}{BMs^2} \exp\left[\frac{-2}{B+C}(y/Ms^2+MH^2)\right] \sum_{r=0}^{\infty}\left(\frac{MsH}{2\sqrt{y}}\,\frac{B-C}{B}\right)^r I_r\left[\frac{4H\sqrt{y}}{s(B+C)}\right],\ y > 0 \qquad (2\text{-}37)$$

$$= \frac{1}{BMs^2} \exp\left[\frac{2y}{Ms^2(B-C)} - \frac{MH^2}{B+C}\left(2 - \frac{B-C}{B}\right)\right],\ y < 0. \qquad (2\text{-}38)$$

2.2.2 THE EXPLICIT PROBABILITY DENSITY FUNCTION FOR THE STANDARD ARRAY MODEL

For the standard array model, from (1-4) we have for the array sum amplitude, $S = NAD(\theta'/N;\ N)$, where we have used $\theta' = \frac{1}{2}N\beta\sin\alpha$, N = total number of hydrophones. As in (2-28, 2-29, 2-30) we may write for the array sum,

$$\sigma^2 = Ns^2 + 2(N-1)as^2$$
$$h^2 = \frac{N^2H^2D^2(\theta'/N;\ N)}{N + 2(N-1)a}$$

ADJACENT CORRELATION (2-39)

$$\sigma^2 = Ns^2 + 2N(N-1)s^2\left(\frac{b+e^{-b}-1}{b^2}\right)$$
$$h^2 = \frac{b^2NH^2D^2(\theta'/N;\ N)}{b^2+2(N-1)(b+e^{-b}-1)}$$

EXPONENTIALLY DECREASING CORRELATION (2-40)

and we may also define $B' = \sigma^2/Ns^2$. (2-41)

The white noise case corresponds to B' =1.

With these definitions, we substitute in (2-25) to find the explicit probability density function for the standard array filter output:

$$p_{sq}(y) = p_{sq}(y;\ \alpha,\ \beta,\ N,\ s;\ H,\ a \text{ or } b)$$

$$= \frac{1}{B'Ns^2} \exp\left\{\frac{-1}{B'Ns^2}(y + N^2H^2D^2s^2)\right\} I_0\left(\frac{2HD}{B's}\sqrt{y}\right),\ y > 0. \qquad (2\text{-}42)$$

2.2.3 COMPUTED RESULTS

Results for the array probability densities (2-31, 2-32) and (2-42) are presented in Figures 2-6 through 2-14.

For the multiplicative array model, the nominal case of 10 elements ($M = 5$), white noise ($a = 0$), unit noise variance ($s^2 = 1$), zero bearing ($\alpha = 0$), half-wavelength array spacing ($\beta = \pi$), and zero dB input SNR ($H^2 = 1$) was chosen. In Figure 2-6, H^2 is varied, yielding curves very similar to those of Figure 2-2, although we are now dealing with an order of magnitude higher values of the filter output, since each array sum is contributed to by five array elements, and the normalization we have chosen is at the elements.

The number of array elements (2M) is varied in Figure 2-7, showing especially well the effect of this parameter on the mean and variance. It is evident that the number of elements does not significantly change the shape of the distribution, as does the SNR. However, for lower input SNR, as in Figure 2-8, a more dramatic effect can be observed.

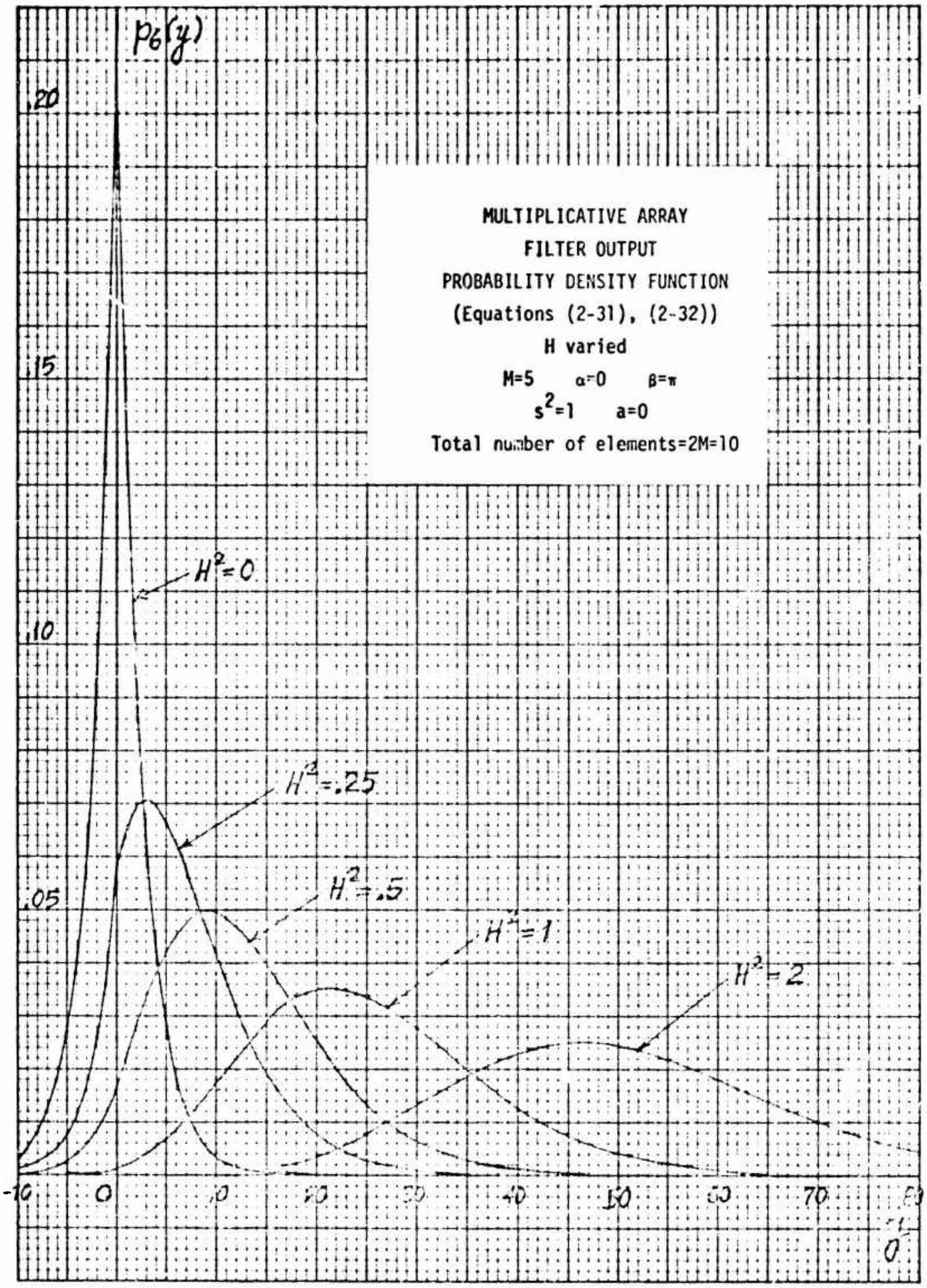


FIGURE 2-6 MULTIPLICATIVE ARRAY PDF;
INPUT SNR VARIED

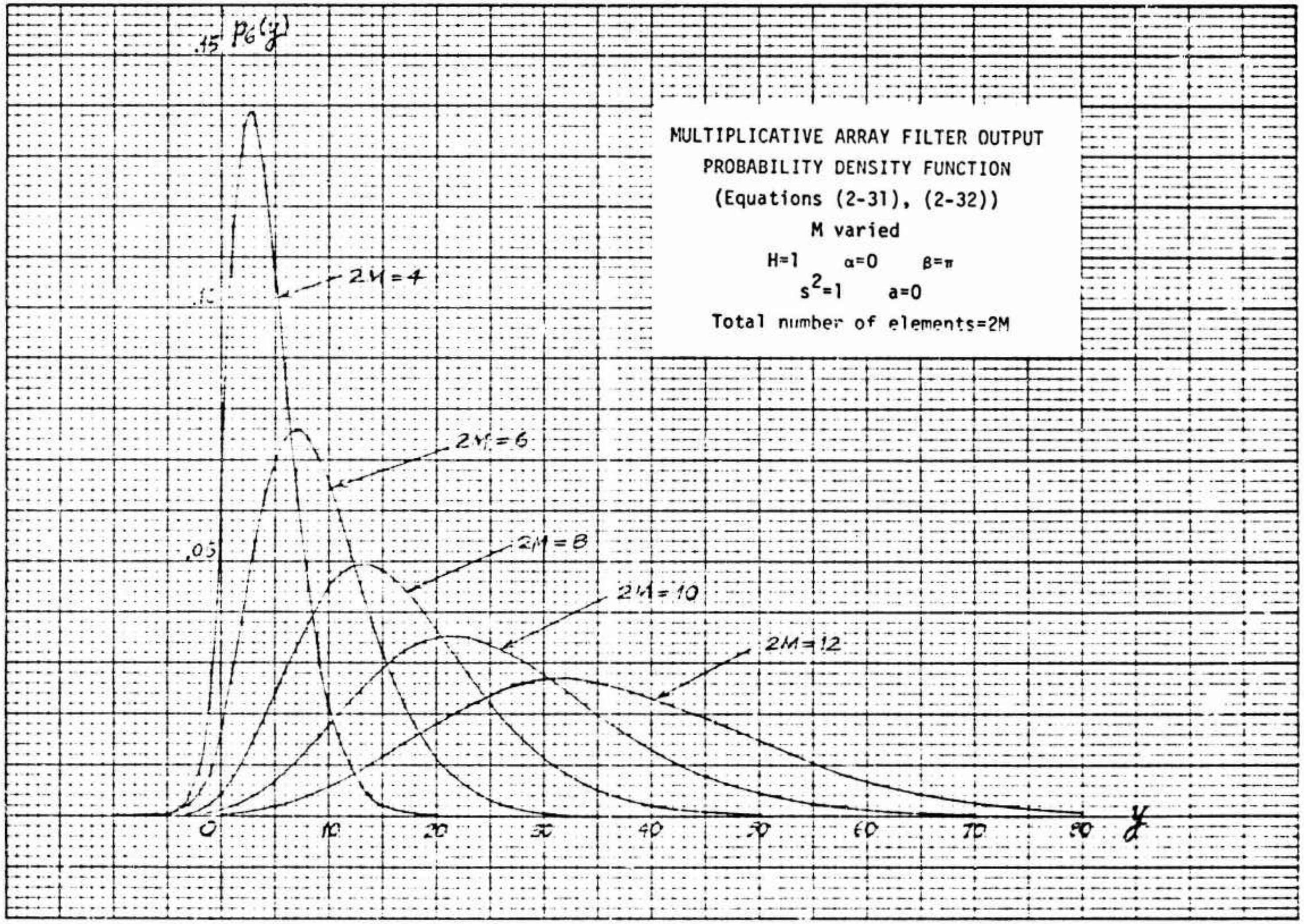


FIGURE 2-7 MULTIPLICATIVE ARRAY PDF; NUMBER OF ELEMENTS VARIED

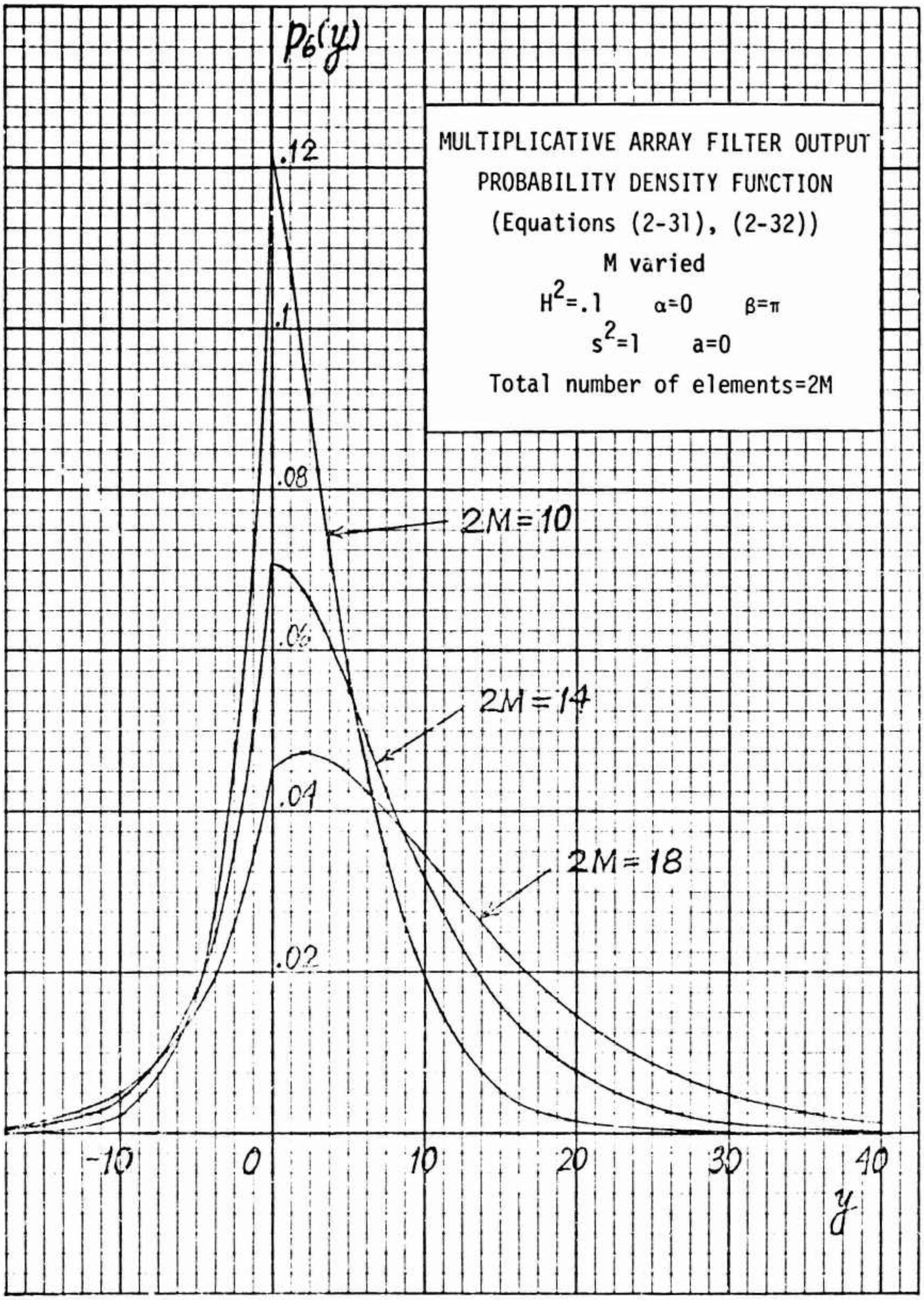


FIGURE 2-8 MULTIPLICATIVE ARRAY PDF;
NUMBER OF ELEMENTS VARIED (LOW SNR)

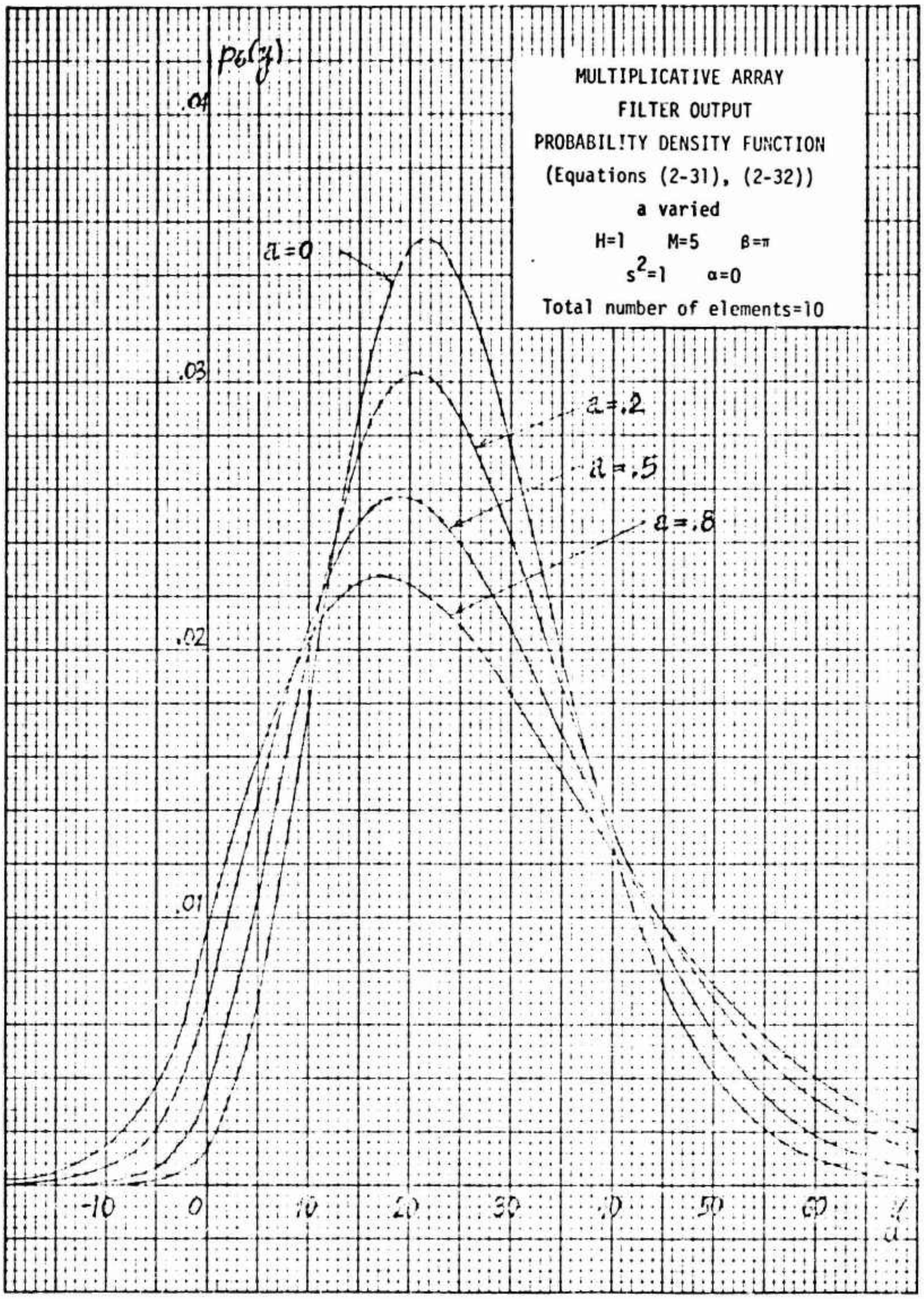


FIGURE 2-9 MULTIPLICATIVE ARRAY PDF;
ADJACENT-ELEMENT CORRELATION VARIED

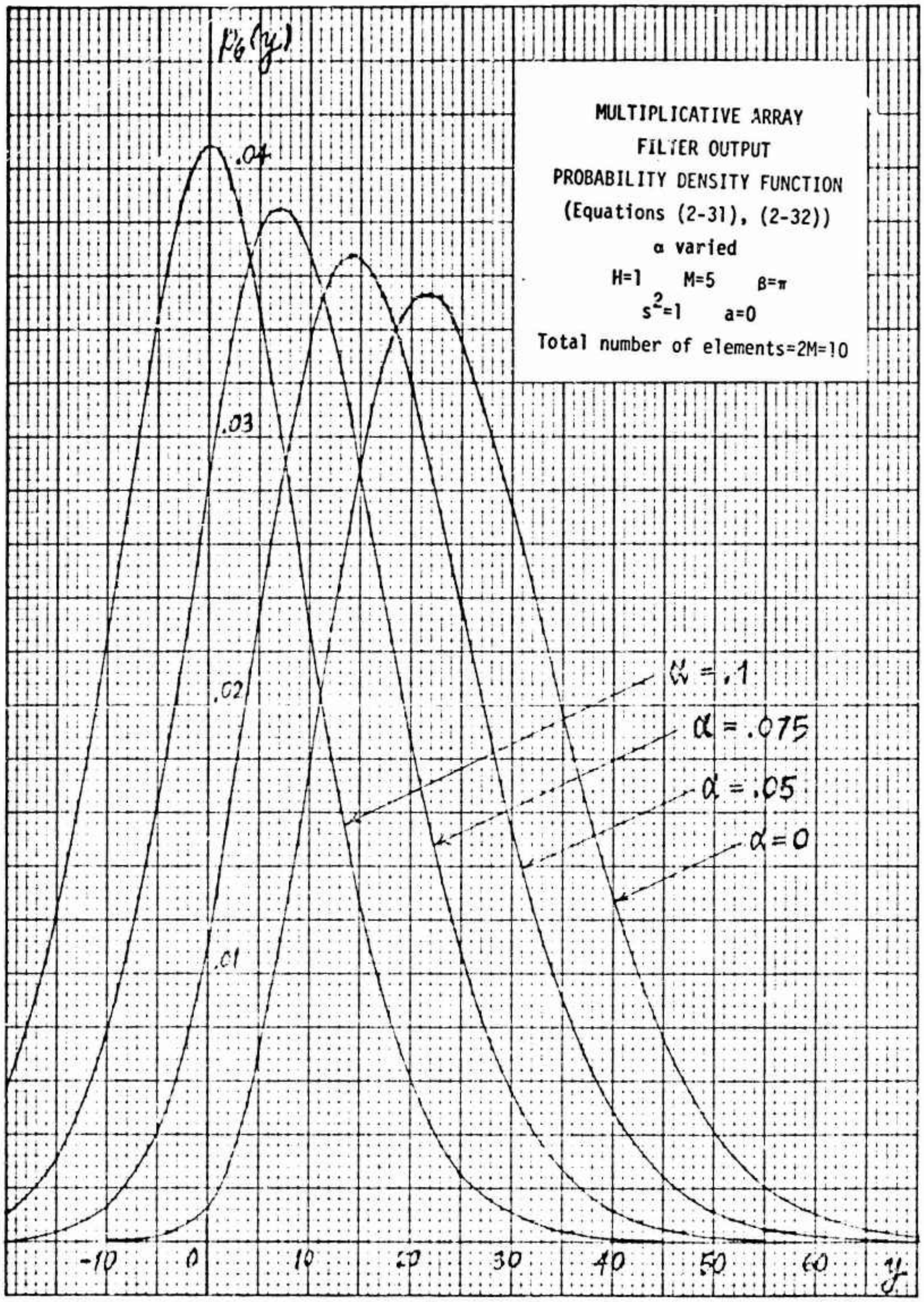


FIGURE 2-10 MULTIPLICATIVE ARRAY PDF; BEARING VARIED

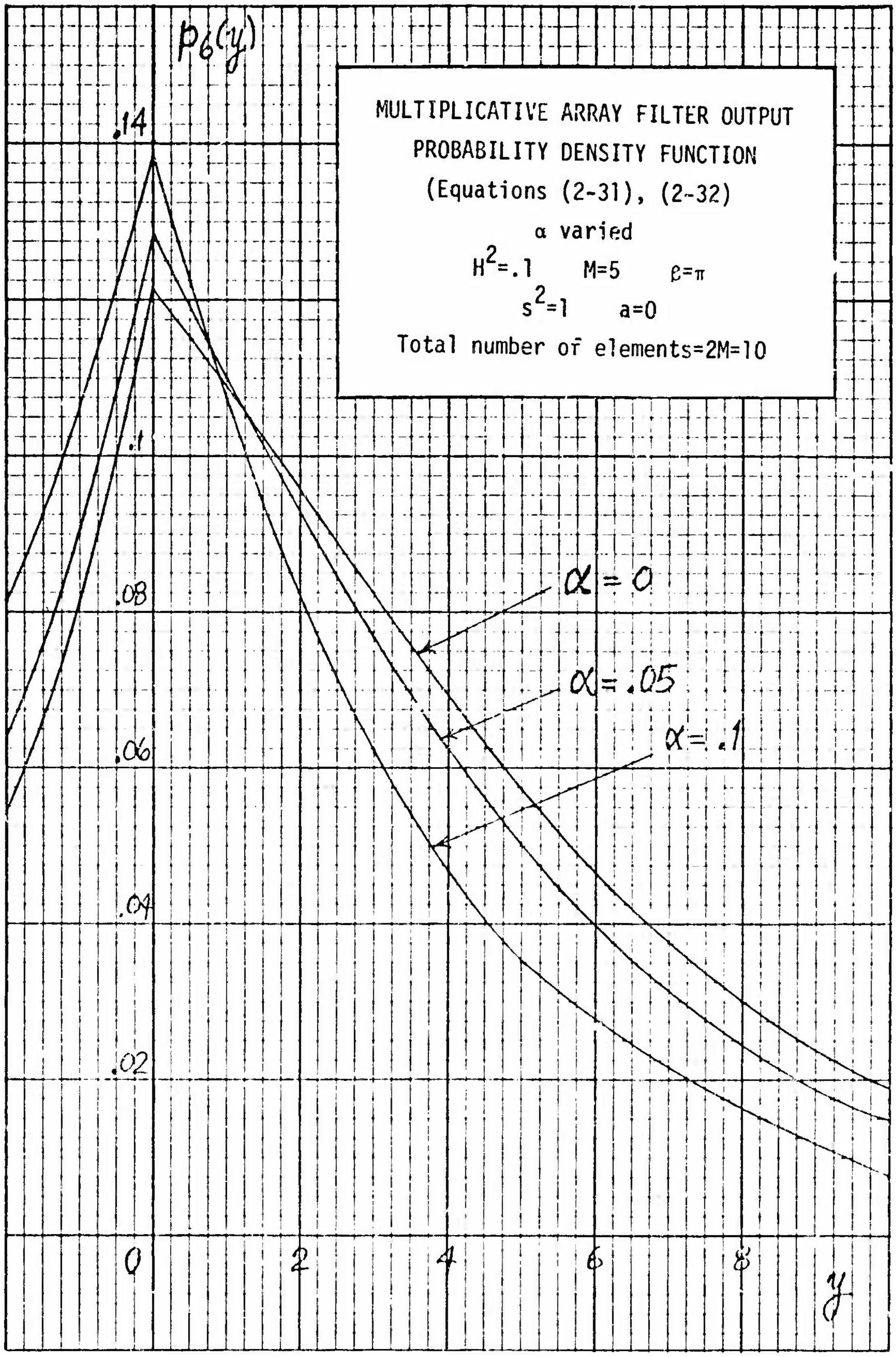


FIGURE 2-11 MULTIPLICATIVE ARRAY PDF;
BEARING VARIED (LOW SNR)

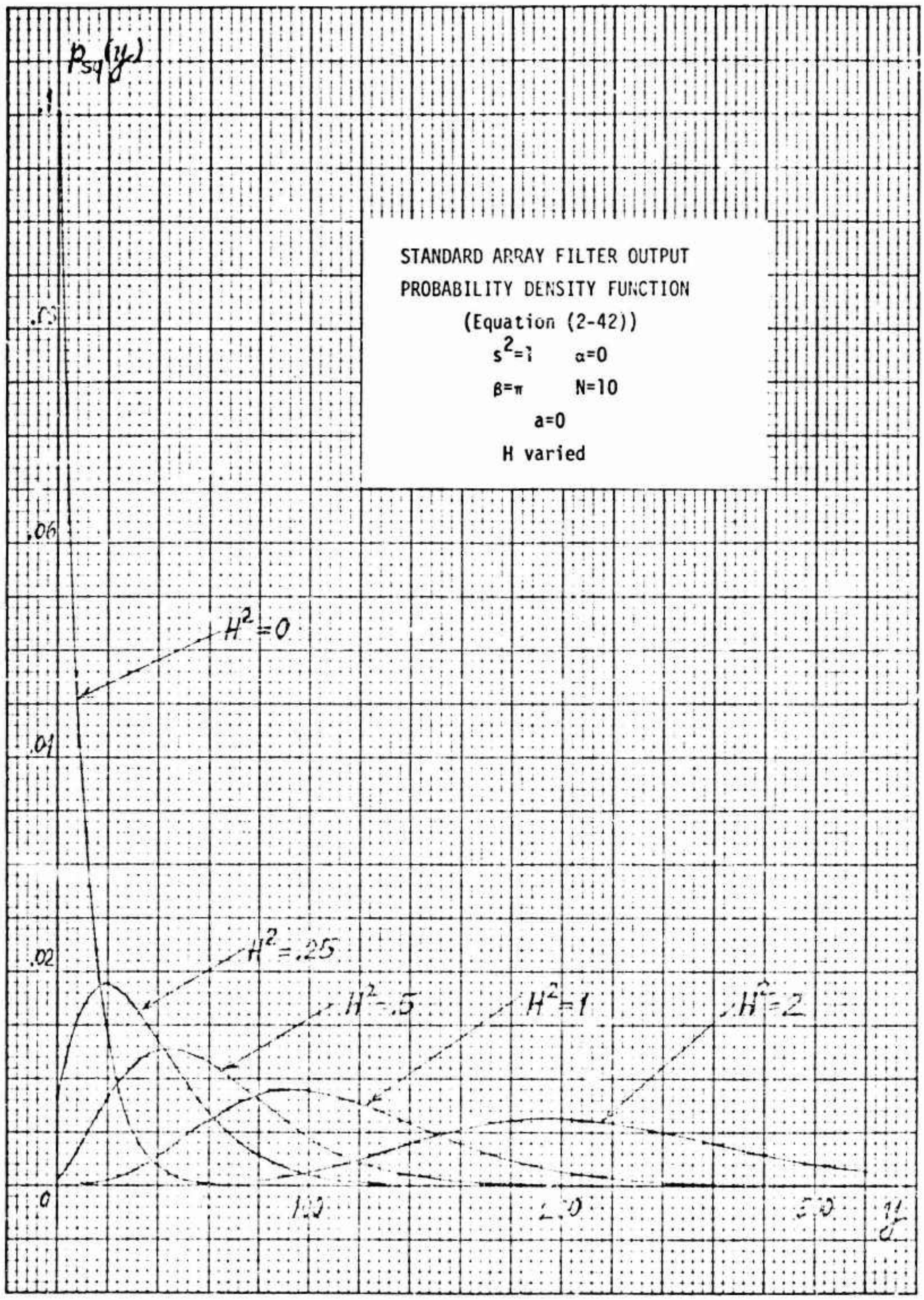


FIGURE 2-12 STANDARD ARRAY PDF; INPUT SNR VARIED

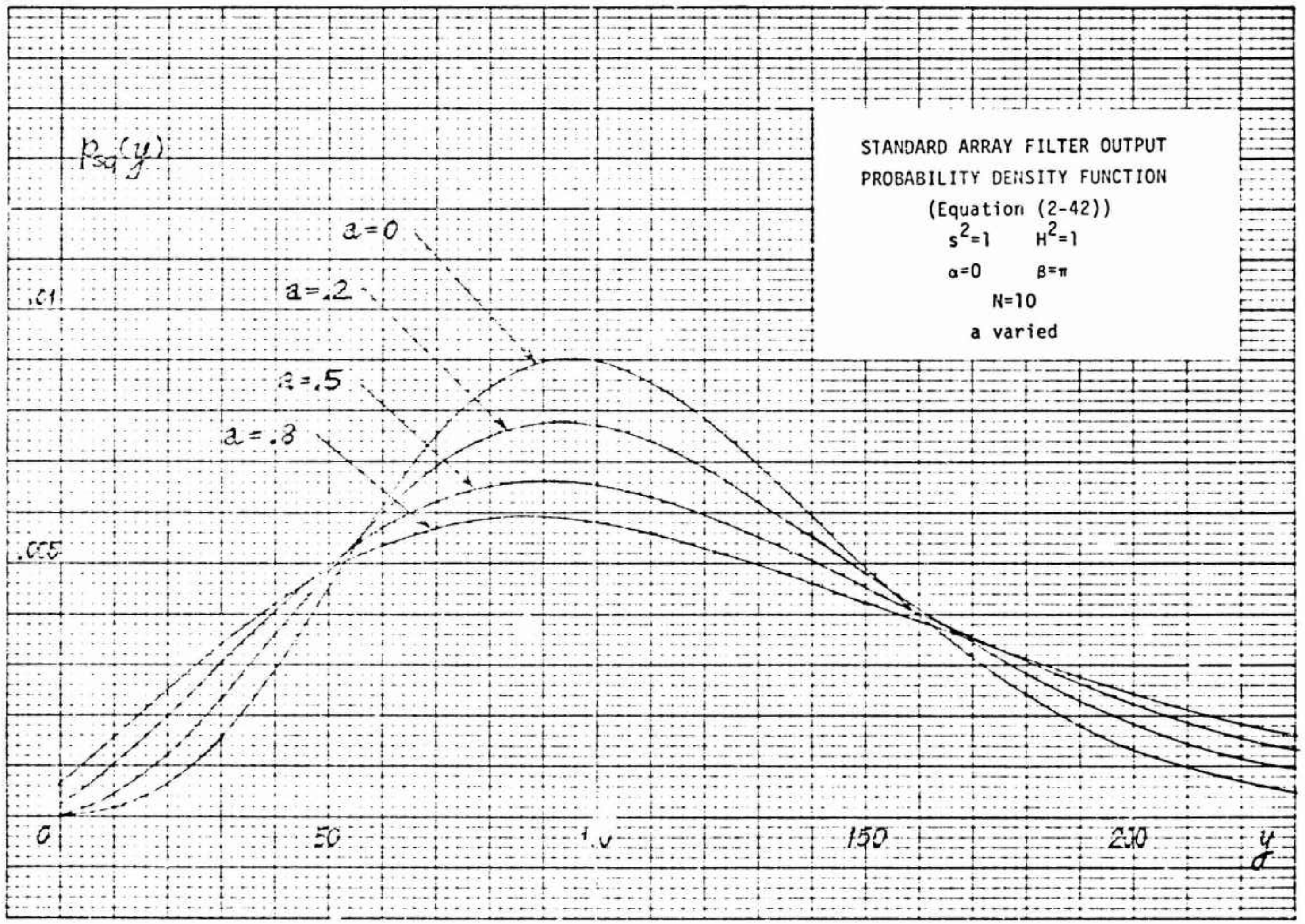


FIGURE 2-13 STANDARD ARRAY PDF; ADJACENT-ELEMENT CORRELATION VARIED

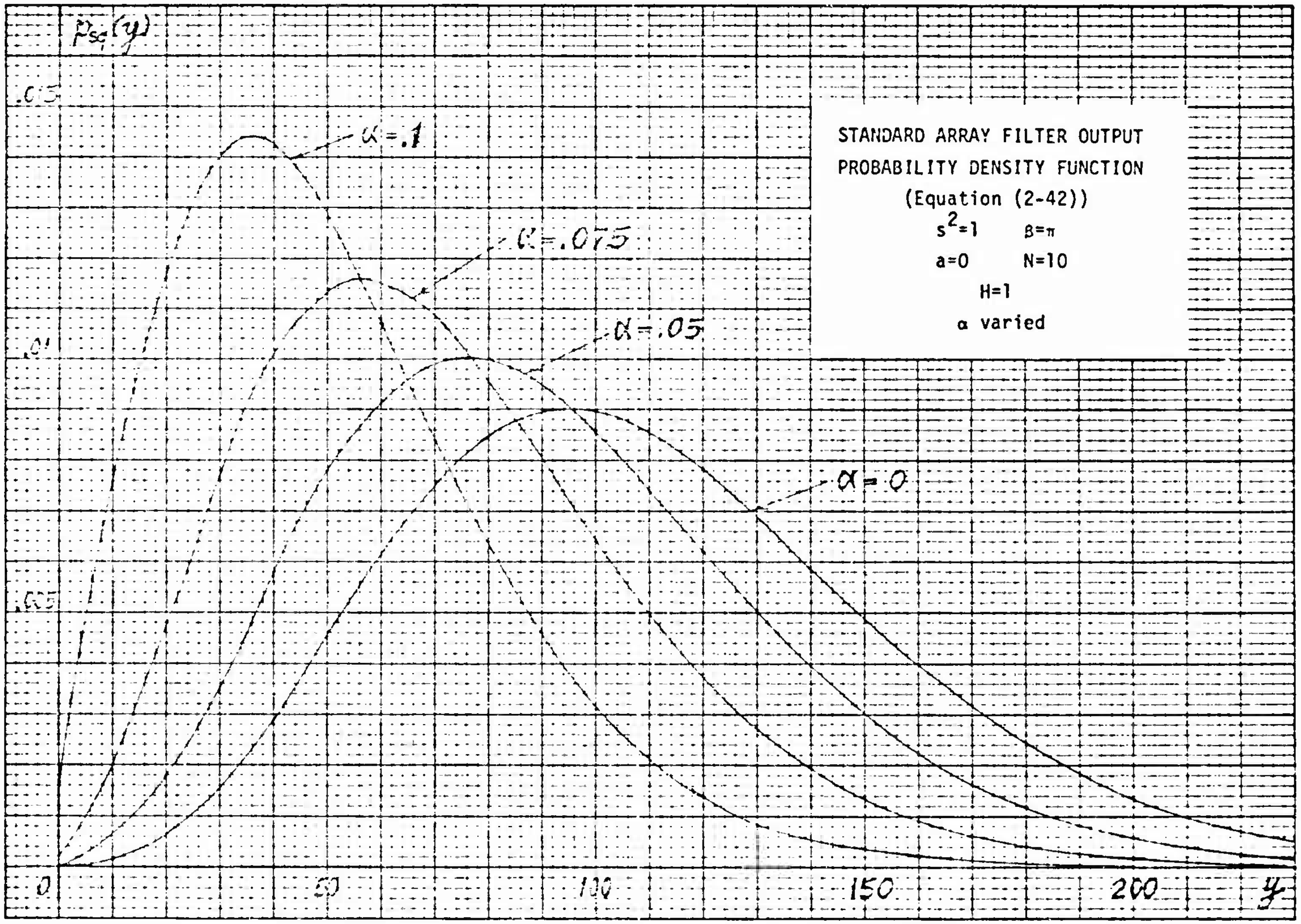


FIGURE 2-14 STANDARD ARRAY PDF; BEARING VARIED

The effect of adjacent inter-element correlation is displayed in Figure 2-9. It is seen that correlation among the array elements, and therefore between the array sums, increases the variance of the filter output and decreases the value of y at which the peak of the distribution occurs.

Figure 2-10 demonstrates a phenomenon of great interest to us in Chapter 4. The bearing is seen to act as a location parameter for the distribution of the filter output. The manner in which the bearing slightly affects the SNR, through the directivity function, can also be observed in this figure. In Figure 2-11 it is shown that for low SNR, bearing ceases to be a location parameter in the usual sense, since the peak of the distribution remains at $y = 0$.

In Figures 2-12 through 2-14, the distribution of the filter output of the square law or standard array model is seen to behave with parameter variation in much the same way as that of the multiplicative. Here we have used $H^2 = 1$, $s^2 = 1$, $\beta = \pi$, $\alpha = 0$, $a = 0$, and $N = 10$ as the nominal case. In Figure 2-10, the SNR is varied, while in Figure 2-13 and 2-14 the array inter-element correlation and the bearing are varied, respectively. It is very noticeable that, although we have used the same number of elements as in the multiplicative case, the magnitude of the filter output values is much greater. In fact, there is a factor of four involved, as can be understood by comparing (1-4) and (1-5) for $N = 2M$.

2.3 MEAN, VARIANCE AND SNR AT THE FILTER OUTPUT

2.3.1 MULTIPLICATIVE ARRAY

For the narrowband multiplicative configuration, we have from Section 2.1,

$$y = (s_1 + n_1)(s_2 + n_2) - (\text{terms of frequency } 2\omega)$$

$$= \tfrac{1}{2}\Big[S_1(n_{2c}\cos\theta_1 + n_{2s}\sin\theta_1) + S_1S_2\cos(\theta_1-\theta_2) + S_2(n_{1c}\cos\theta_2 + n_{1s}\sin\theta_2)$$

$$+ n_{1c}n_{2c} + n_{1s}n_{2s}\Big] \qquad (2\text{-}43)$$

so that $E(y) = \rho\sigma_1\sigma_2 + \tfrac{1}{2}S_1S_2\cos 2\theta = \sigma_1\sigma_2(\rho + h_1h_2\cos 2\theta)$ (2-44)

and $\mathrm{Var}(y) = \tfrac{1}{2}\sigma_1^2\sigma_2^2(1+\rho^2-r^2) + \tfrac{1}{2}\sigma_1\sigma_2S_1S_2(\rho\cos 2\theta + r\sin 2\theta) + \tfrac{1}{4}(\sigma_1^2S_2^2 + \sigma_2^2S_1^2)$

$$= \tfrac{1}{2}\sigma_1^2\sigma_2^2[1 + \rho^2 - r^2 + 2h_1h_2(\rho\cos 2\theta + r\sin 2\theta) + h_1^2 + h_2^2] \qquad (2\text{-}45)$$

where $\theta = \frac{1}{2}(\theta_1-\theta_2)$ and we have used (1-7) and the well known moment properties of zero-mean Gaussian random variables ([16], Section 7.3.2):

$$E(uvwx) = E(uv)E(wx) + E(uw)E(vx) + E(ux)E(vw) \qquad (2\text{-}46)$$

$$E(uvw) = 0 \qquad (2\text{-}47)$$

with (u, v, w, x) being any combination of $(n_{1c}, n_{1s}, n_{2c}, n_{2s})$.

For our main case of the symmetric array, even noise spectrum, we have $\sigma_1 = \sigma_2 = \sigma$, $S_1 = S_2 = \sigma h\sqrt{2}$, and $r = 0$. Therefore, for the main case,

$$E(y) = \sigma^2(\rho + h^2 \cos 2\theta) \tag{2-48}$$

$$\mathrm{Var}(y) = \tfrac{1}{2}\sigma^4[1 + \rho^2 + 2h^2(1 + \rho \cos 2\theta)]. \tag{2-49}$$

Further, using the notation of (2-29), 2-30, 2-31), we obtain explicitly

$$E(y) = Ms^2[C + MH^2D^2(\theta/M;\ M) \cos 2\theta] \tag{2-50}$$

$$\mathrm{Var}(y) = \tfrac{1}{2} M^2 s^4 [B^2 + C^2 + MH^2D^2(\theta/M)(2B + C \cos 2\theta)]. \tag{2-51}$$

We may use these expressions to show the SNR at the filter output, using the conventions of Middleton[16], Section 5.3:

$$SNR_0 = \frac{[E(y)-E(y|H=0)]^2}{\mathrm{Var}(y)} = \frac{2M^2H^4D^4\cos^2 2\theta}{B^2 + C^2 + MH^2D^2(2B + C\cos 2\theta)}. \tag{2-52}$$

2.3.2 SQUARE-LAW ARRAY

For the narrowband standard or square law array configuration, the filter output is a noncentral chi-square random variable, as noted in Section 2.1.4. Since the mean and variance of $\chi^2(x;\ 2,\ a^2)$ are given ([23]) by $E(x) = 2 + a^2$, $\mathrm{Var}(y) = 4 + 4a$, we have

$$E(y) = \sigma^2(1 + h^2) \tag{2-53}$$

$$\mathrm{Var}(y) = \sigma^4(1 + 2h^2). \tag{2-54}$$

Using the notation of (2-39, 2-40, 2-41), we write explicitly,

$$E(y) = Ns^2[B' + NH^2D^2(\theta'/N;\ N)] \tag{2-55}$$

$$\mathrm{Var}(y) = B'N^2s^4[B' + 2NH^2D^2(\theta'/N;\ N)]. \tag{2-56}$$

The filter output SNR then becomes

$$SNR_o = \frac{[E(y) - E(y\ H=0)]^2}{Var(y)} = \frac{N^2H^4D^4}{B'(B' + 2NH^2D^2} . \tag{2-57}$$

2.3.3 COMPARISONS

It is interesting to compare the moments and SNRs which we have just shown. We shall do so for the white noise case in which also $\theta = 0$ (boresight case). Denoting means by E and variances by V, we have

$$\begin{aligned}
E_{sq} &= Ns^2(1 + NH^2) & E_m &= M^2s^2H^2 \\
V_{sq} &= N^2s^4(1 + 2NH^2) & V_m &= \tfrac{1}{2}M^2s^4(1 + 2MH^2) \\
SNR_{sq} &= \frac{N^2H^4}{1 + 2NH^2} & SNR_m &= \frac{2M^2H^4}{1 + 2MH^2}
\end{aligned} \tag{2-58}$$

The comparisons we wish to show are tabulated as Figure 2-15, in which we have four situations representing high and low input SNR for (a) the standard and multiplicative arrays having the same total number of elements, that is, N = 2M; (b) the two arrays having roughly the same beamwidth, represented by N = 4M. It is seen, for example, that the multiplicative array SNR is less than or equal to that for the standard array in all these cases.

* * * * * *

Additional references. The method we have used to obtain the filter output probability density functions is by no means the only one. Perhaps the most thorough method is that which takes into account the filter transfer function of a realizable filter, and treats the filter

	Same Number of Elements $N = 2M$	Same Beamwidth $N = 4M$
Low SNR	$E_m = \frac{NH^2}{4} E_{sq}$ $V_m = \frac{1}{8} V_{sq}$ $SNR_m = \frac{1}{4} SNR_{sq}$	$E_m = \frac{NH^2}{16} E_{sq}$ $V_m = \frac{1}{32} V_{sq}$ $SNR_m = \frac{1}{8} SNR_{sq}$
High SNR	$E_m = \frac{1}{4} E_{sq}$ $V_m = \frac{1}{16} V_{sq}$ $SNR_m = SNR_{sq}$	$E_m = \frac{1}{16} E_{sq}$ $V_m = \frac{1}{128} V_{sq}$ $SNR_m = \frac{1}{2} SNR_{sq}$

FIGURE 2-15 COMPARISON OF FILTER OUTPUT MOMENTS, SNR

input as a time series. In the literature, this approach is often attributed to Kac and Siegert[40], and usually is found applied to the square law detector [41-43]. An exception is the work of Lampard[15], who studies a multiplier/filter configuration. The curves in Marcum[30] and in Emerson[41] are especially comparable to those we have generated.

Others not mentioned in the text who have used the same method as we to analyze the multiplier/filter, though less generally, are listed for reference [44-46]. A thorough treatment of the output SNR of such systems is that of Green[47]. In these works, very similar probability density curves were obtained from sometimes very dissimilar expressions.

For those who wish to pursue the mathematical statistics aspect of this problem, we provide a listing [48-52] of what we consider profitable reading in this area.

CHAPTER THREE

DETECTION PERFORMANCE

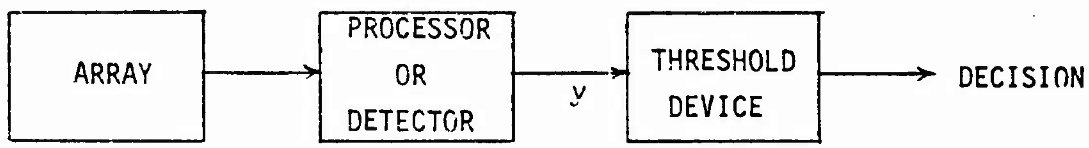
Having now the probability density functions for the array processor filter outputs, we are in a position to evaluate what Tucker[5] described as "... the difficult question of whether the better signal/noise performance [of one detector over another] is really indicative of a higher probability of detection." That is, we are able to calculate directly the probabilities of detection for the standard and multiplicative array models, as functions of the SNR and of corresponding probabilities of false alarm.

3.1 DETECTION CRITERIA

The above assertion relies on various definitions and assumptions. Our analysis assumes the usual Neyman-Pearson arrangement[26], shown in Figure 3-1, in which the value of the filter output is subjected to a "test." If the filter output (y) is greater than a threshold (τ), then the decision, "signal present," is made. If $y < \tau$, the decision "no signal" is made.

Since due to the noise, y is a random variable, this decision strategy is susceptible to two kinds of error. One, called "false alarm," occurs when a threshold-crossing due to noise only causes a spurious "signal present" decision. The other error lies in missing a weak signal because a threshold-crossing did not take place.

The problem in this type of strategy is to choose the threshold such that an acceptable probability of error is maintained. Often there is a high cost attached to a false alarm (for example, an expensive weapon fired), so the usual practice is to consider the threshold as $\tau = \tau(P_{FA})$ where P_{FA} is the probability of false alarm.



DECISION: "Signal present" when $y > \tau$

"No signal" when $y \leq \tau$

FIGURE 3-1 SIGNAL DETECTION METHOD

Various types of detection systems then are evaluated in terms of the probability of detection (P_D) achieved for a given threshold, as a function of the SNR when a signal is present. Therefore the tradeoff between P_{FA} and P_D is of great interest, and it is standard practice to exhibit this tradeoff in the form of "receiver operating characteristics," in which P_D is plotted as a function of P_{FA} and the input SNR.

3.2 CALCULATION OF FALSE ALARM PROBABILITY

Since a false alarm occurs when the filter output y exceeds the detection threshold τ when there is no signal present, the probability of false alarm is given by

$$P_{FA} = \int_{\tau}^{\infty} p(y|H=0)\,dy. \tag{3-1}$$

3.2.1 MULTIPLICATIVE MODEL

Using (2-26) for $p_0(y|H=0)$, we have for the false alarm probability for the multiplicative array model,

$$P_{FA} = \int_{\tau}^{\infty} dy\, \frac{1}{\sigma^2} \exp\left\{\frac{-2y}{\sigma^2(1+\rho)}\right\} \qquad (\tau>0)$$

$$= \tfrac{1}{2}(1+\rho) \int_{\frac{2\tau}{\sigma^2(1+\rho)}}^{\infty} dx\, e^{-x}$$

$$P_{FA} = \tfrac{1}{2}(1+\rho) \exp\left\{\frac{-2\tau}{\sigma^2(1+\rho)}\right\} \tag{3-2}$$

or, to use the notation of Section 2.2.1,

$$P_{FA} = \tfrac{1}{2}(1+C/B) \exp\left\{\frac{-2\tau}{Ms^2(B+C)}\right\}. \tag{3-3}$$

3.2.2 SQUARE LAW MODEL

Using (2-26) for $p_{sq}(y)$, we obtain for the false alarm probability for the square law or standard array model,

$$P_{FA} = \int_{\tau}^{\infty} dy \, \frac{1}{\sigma^2} \exp(-y/\sigma^2) \qquad (\tau>0)$$

$$= \exp(-\tau/\sigma^2) \tag{3-4}$$

or, in terms of Section 2.2.2,

$$P_{FA} = \exp(-\tau/B'Ns^2). \tag{3-5}$$

3.2.3 COMPUTED RESULTS

Figure 3-2 illustrates the simple manner in which the false alarm probability varies with the threshold. It is clear that the standard array model requires a higher detection threshold than the multiplicative with the same number of elements to achieve a similar false alarm rate. However, this statement has to be qualified further for general application, since the models do not take into account the attenuations and gains of actual systems, and therefore the magnitudes of the filter outputs may not correspond in the same way as they do here.

Nevertheless, the mean and variance of the filter output due to noise alone tends to be smaller in the multiplicative configuration, allowing a relatively lower threshold (compare equations (2-51) and (2-56), (2-52) and (2-57) for N = 2M).

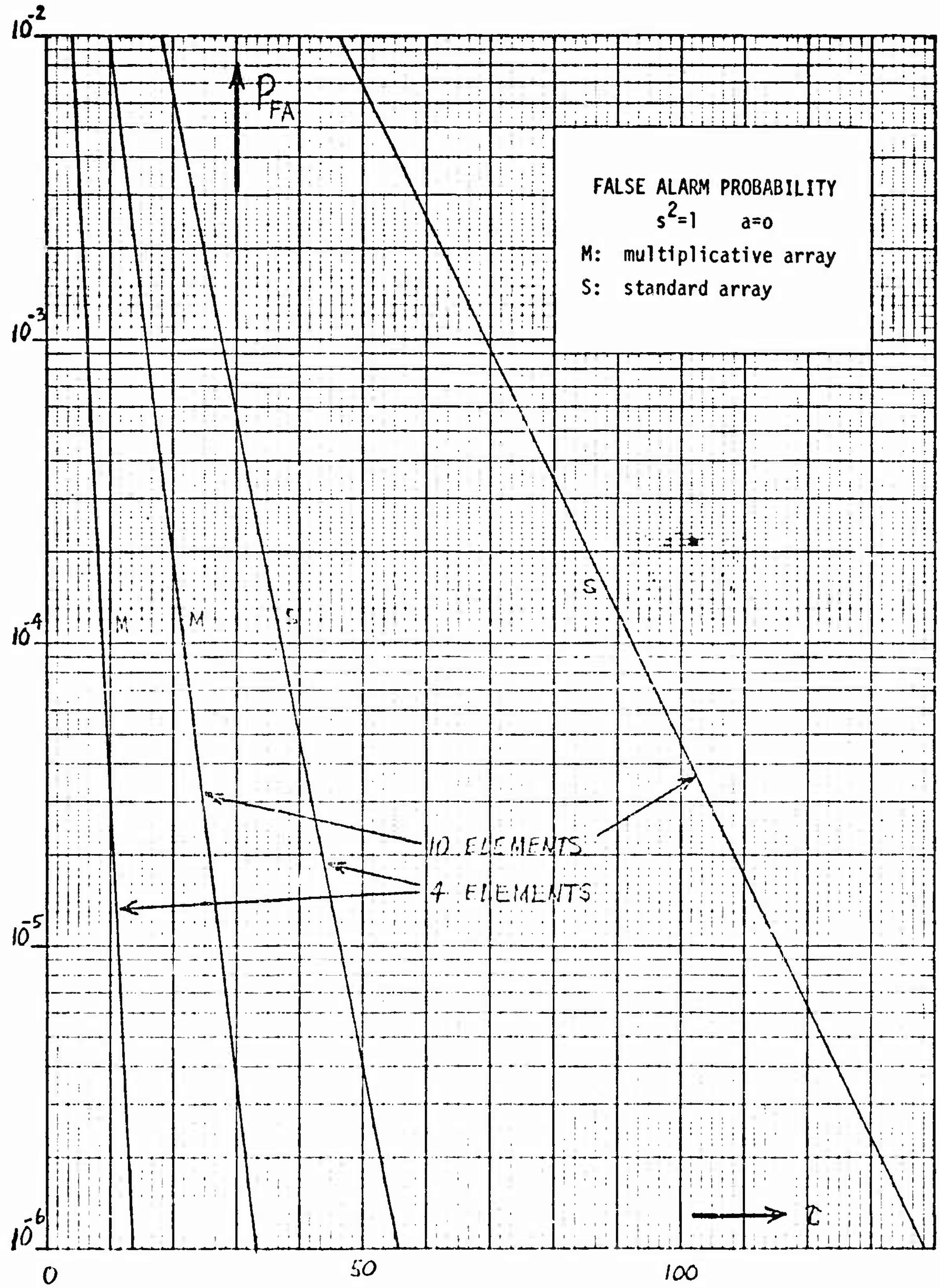


FIGURE 3-2 FALSE ALARM PROBABILITY vs THRESHOLD

3.3 CALCULATION OF DETECTION PROBABILITY

Detection of a sigral which is present occurs when the filter output exceeds the detection threshold. Therefore, the probability of detection is given by

$$P_D = \int_{\tau}^{\infty} dy\ p(y|H>0). \tag{3-6}$$

3.3.1 MULTIPLICATIVE MODEL

Using (2-26) for $p_6(y)$, for the multiplicative array model we have for the probability of detection,

$$p_6(y) = \frac{1}{\sigma^2} \exp\left[-(h_3^2 + h_4^2)\right] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left[\frac{1}{2}(1-\rho)h_3^2\right]^m}{m!} \frac{\left[\frac{1}{2}(1+\rho)h_4^2\right]^n}{n!}$$

$$\times \int_{\tau}^{\infty} dy \exp\left\{\frac{-2y}{\sigma^2(1+\rho)}\right\} G_m^n\left[\frac{4y}{\sigma^2(1-\rho^2)}\right]. \quad (\tau>0). \tag{3-7}$$

To obtain the result, we take the integral in (3-7) separately:

$$\int_{\tau}^{\infty} dy\ e^{-by} G_m^n(cy) = \sum_{k=0}^{m} \binom{m+n-k}{n} \int_{\tau}^{\infty} dy\ e^{-by} (cy)^k/k!$$

$$= \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{1}{bk!} \left(\frac{c}{b}\right)^k \int_{b\tau}^{\infty} dx\ e^{-x} x^k$$

$$= \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{c^k}{k! b^{k+1}} \Gamma(k+1, b\tau), \tag{3-8}$$

where $\Gamma(n,x)$ is the incomplete Gamma function. Using [13], formula 8.352.2, we continue:

$$\int_{\tau}^{\infty} dy\, e^{-by} G_m^n(cy) = \frac{e^{-b\tau}}{b} \sum_{k=0}^{m} \binom{m+n-k}{n} \left(\frac{c}{b}\right)^k \sum_{r=0}^{k} \frac{(b\tau)^r}{r!}$$

$$= \frac{e^{-b\tau}}{b} \sum_{k=0}^{m} \binom{m+n-k}{n} \left(\frac{c}{b}\right)^k e_k(b\tau) \qquad (3\text{-}9)$$

where we have taken $b = 2/\sigma^2(1+\rho)$ and $c = 4/\sigma^2(1-\rho^2)$.

Substituting (3-9) in (3-7), we find for $h_1=h_2=h$,

$$P_D = \tfrac{1}{2}(1+\rho)\exp\left\{-2h^2\left(\frac{1-\rho\cos 2\theta}{1-\rho^2}\right)\right\} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(h\cos\theta)^{2m}}{m!}\frac{(h\sin\theta)^{2n}}{n!}\left(\frac{1-\rho}{1+\rho}\right)^{m-n}$$

$$\times \exp\left\{\frac{-2\tau}{\sigma^2(1+\rho)}\right\} \sum_{k=0}^{m} \binom{m+n-k}{n}\left(\frac{2}{1-\rho}\right)^k e_k\left\{\frac{2\tau}{\sigma^2(1+\rho)}\right\} \qquad (3\text{-}10)$$

or, in the notation of Section 2.2.1,

$$P_D = \frac{B+C}{2B}\exp\left\{\frac{-2\tau}{Ms^2(B+C)} - \frac{2MH^2D^2}{B}\,\frac{B-C\cos 2\theta}{B^2-C^2}\right\} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{MH^2D^2}{B}\right)^{m+n}\frac{(\cos\theta)^{2m}}{m!}\frac{(\sin\theta)^{2n}}{n!}$$

$$\times \left(\frac{B-C}{B+C}\right)^{m-n} \sum_{k=0}^{m}\binom{m+n-k}{n}\left(\frac{2B}{B-C}\right)^k e_k\left\{\frac{2\tau}{Ms^2(B+C)}\right\}. \qquad (3\text{-}11)$$

3.3.2 SQUARE LAW MODEL

Using (2-25) for $p_{sq}(y)$, the probability of detection for the square law or standard array model becomes

$$P_D = \int_{\tau}^{\infty} dy \frac{1}{\sigma^2} \exp(-h^2 - y/\sigma^2) \; I_0(2h\sqrt{y}/\sigma)$$

$$= \int_{\sqrt{2\tau}/\sigma}^{\infty} x \exp(-h^2 - \tfrac{1}{2}x^2) \; I_0(\sqrt{2}hx) \, dx \tag{3-12}$$

or $$P_D = Q(\sqrt{2}h, \sqrt{2\tau}/\sigma), \tag{3-13}$$

where Q is Marcum's Q-function. In terms of Section 2.2.2, the probability of detection is

$$P_D = Q(HD\sqrt{2N/B'}\,, \sqrt{2\tau/B'Ns^2}). \tag{3-14}$$

3.3.3 COMPUTED RESULTS

The behavior of the probability of detection at the filter outputs as a function of threshold and array input SNR is shown in Figure 3-3 for N = 2M = 4 and in Figure 3-4 for N = 2M = 10.

It is seen that in general, the standard array model's detection probability is higher than that of the multiplicative for the same threshold, subject to the qualifications cited in Section 3.2.3. This is due to the fact that, for the same number of array elements, the bulk of the standard probability density is concentrated about a higher mean value than for the multiplicative case. Another factor is that the detection probability for the square-law or standard processor is always unity for zero threshold.

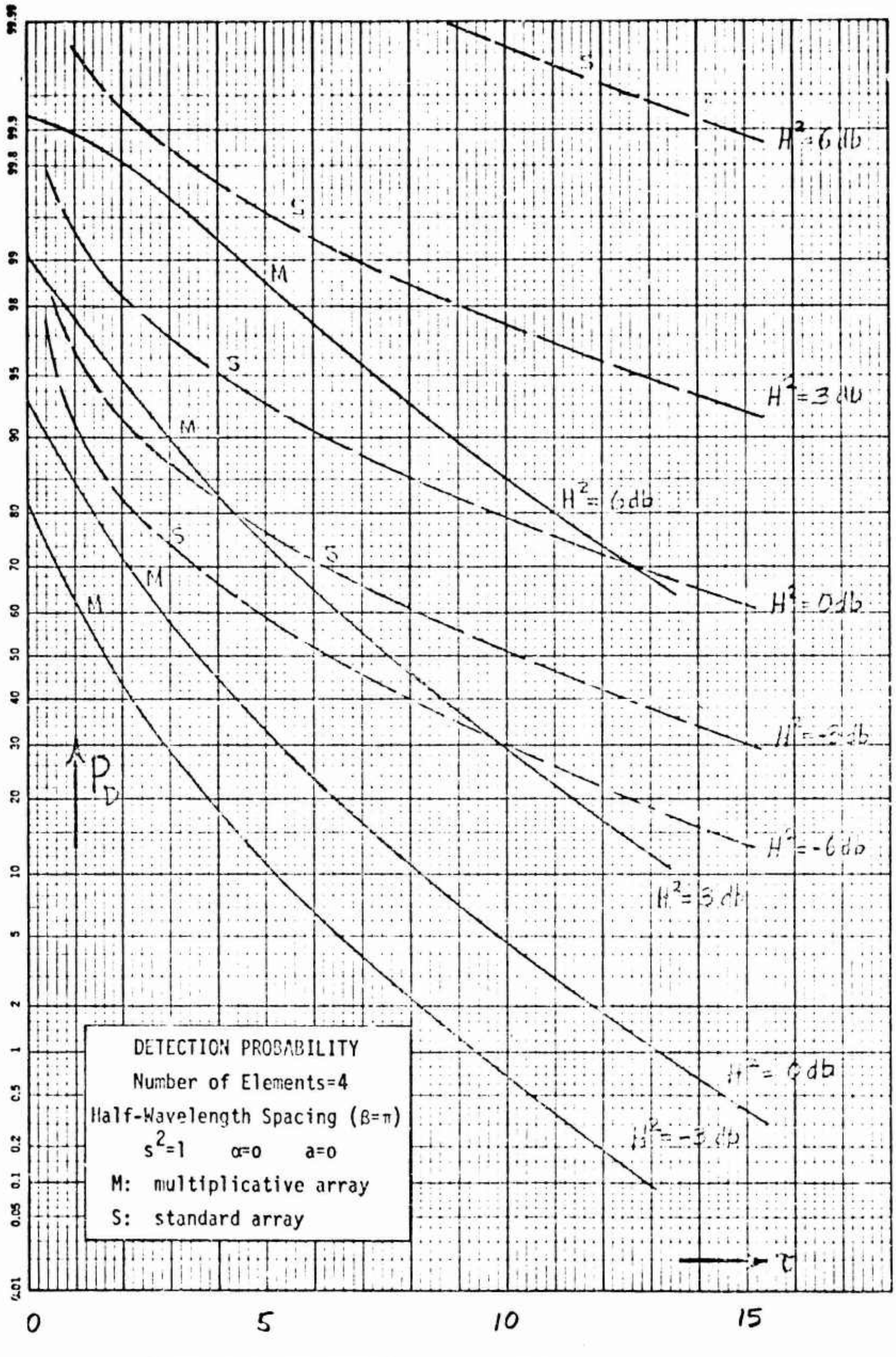


FIGURE 3-3 DETECTION PROBABILITY vs THRESHOLD
(4-ELEMENT ARRAYS)

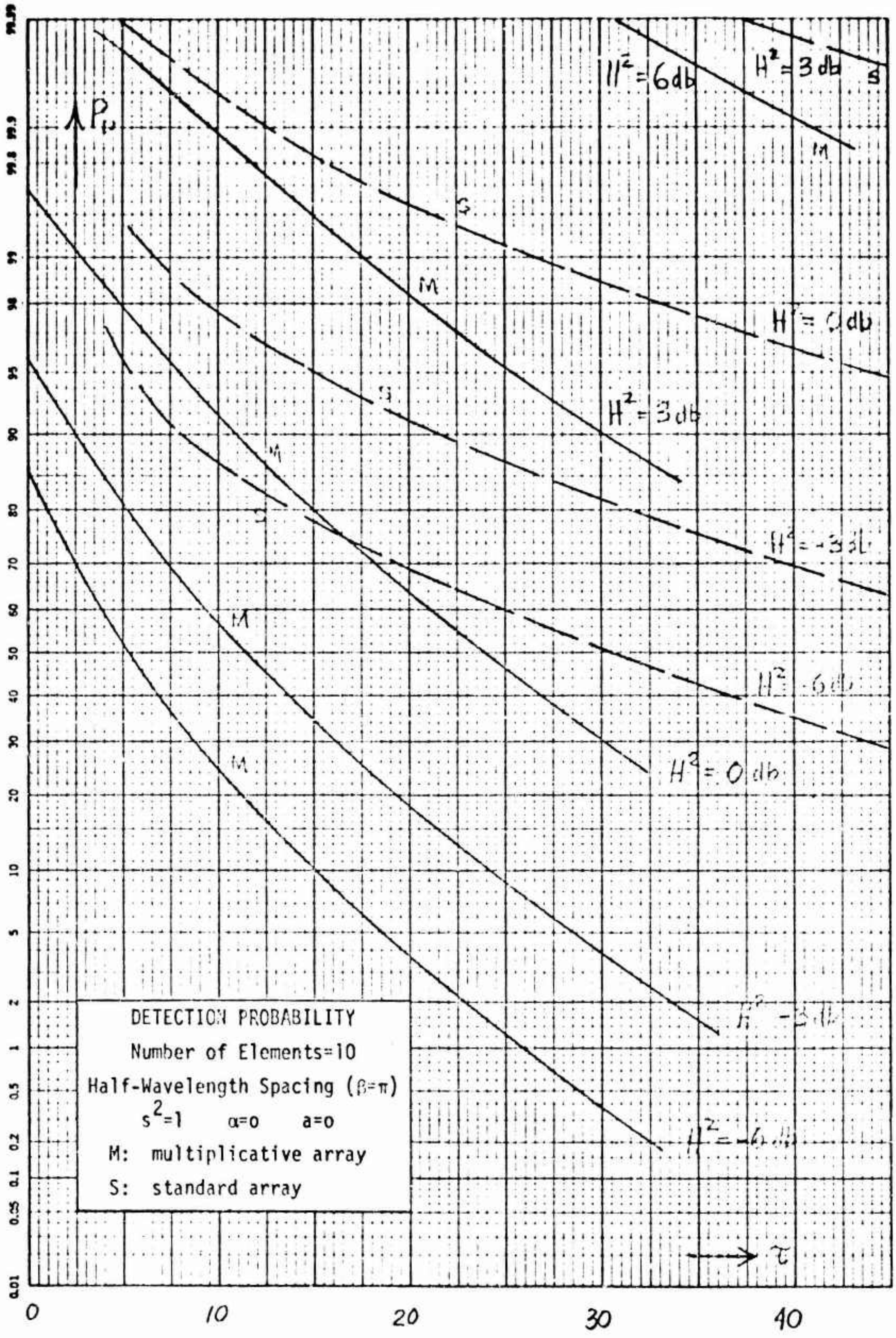


FIGURE 3-4 DETECTION PROBABILITY vs THRESHOLD
(10-ELEMENT ARRAYS)

3.4 SYSTEM COMPARISON: RECEIVER OPERATING CHARACTERISTICS

In the previous sections we have examined separately the probabilities of false alarm and of detection for the two array models we are considering. It was seen that the multiplicative model has a more desirable false alarm rate for the same value of detection threshold, while the square-law or standard model yields a higher probability of detection. However, this apparent tradeoff is blurred by the unsuitability of the detection threshold as a basis for comparison.

A more informative method of comparing such systems is the construction of "receiver operating characteristics," which eliminates the threshold altogether by converting the parametric relations

$$\begin{aligned} P_{FA} &= P_{FA}(\tau) \\ P_D &= P_D(\tau, SNR) \end{aligned} \tag{3-15}$$

into the relationship $P_D = P_D(P_{FA}, SNR)$. (3-16)

Because with the receiver operating characteristics (ROC) representation two different threshold detectors are more directly comparable, a performance index such as "minimum detectable signal" can be used to pronounce one the better detector with some confidence in the generality of this kind of statement. (The minimum detectable signal is defined as the SNR required to insure a given probability of detection for a certain false alarm rate, also given.) Helstrom[26] cites additional uses of the ROC; for example, determination of parameters for Bayes and minimax criteria detection schemes.

The ROC curves for our two models, for the white noise, boresight case, are given in Figure 3-5 for N = 2M = 4 and in Figure 3-6 for N = 2M = 10. The fact that the standard array processor ROC curves

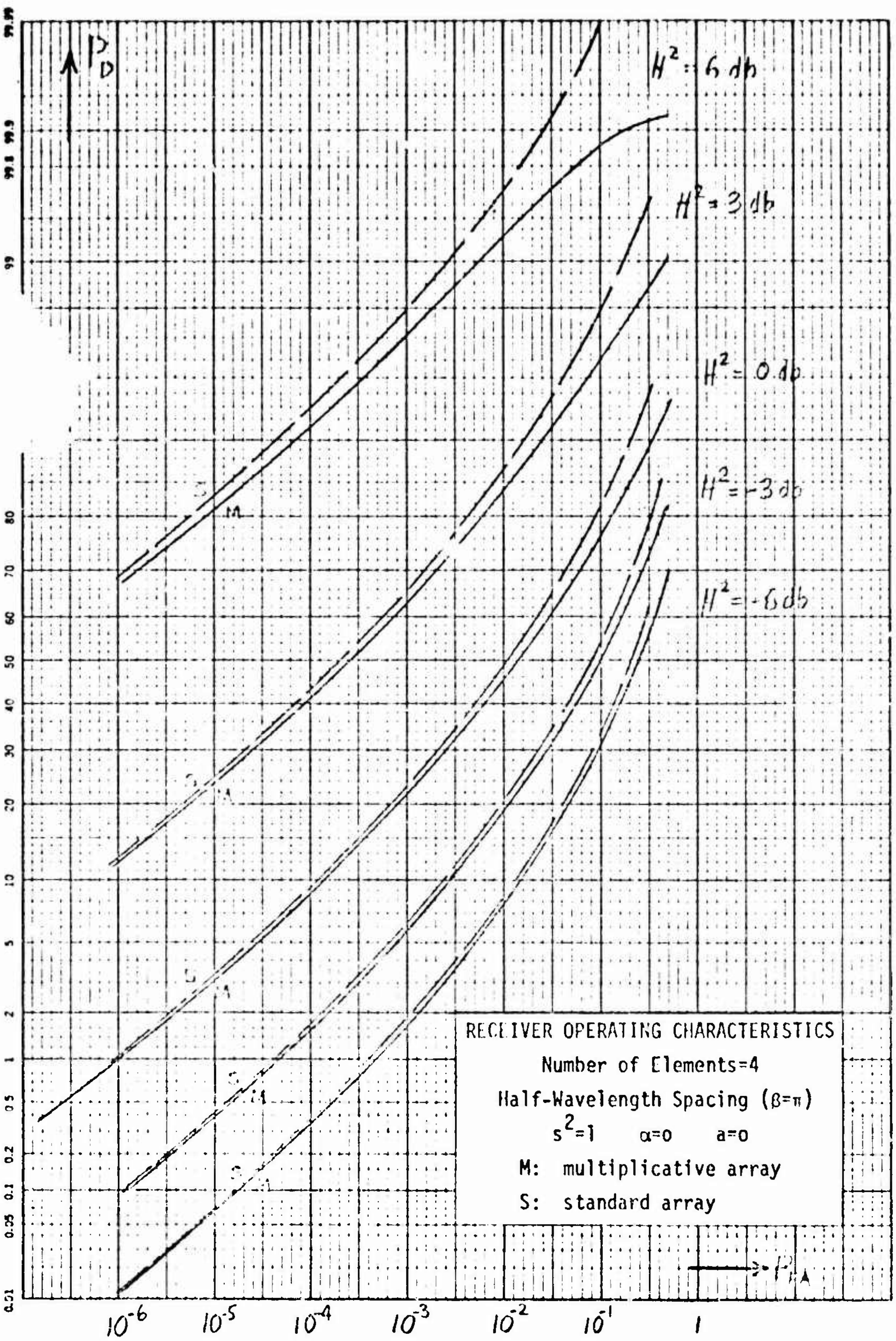


FIGURE 3-5 ROC FOR 4-ELEMENT ARRAYS

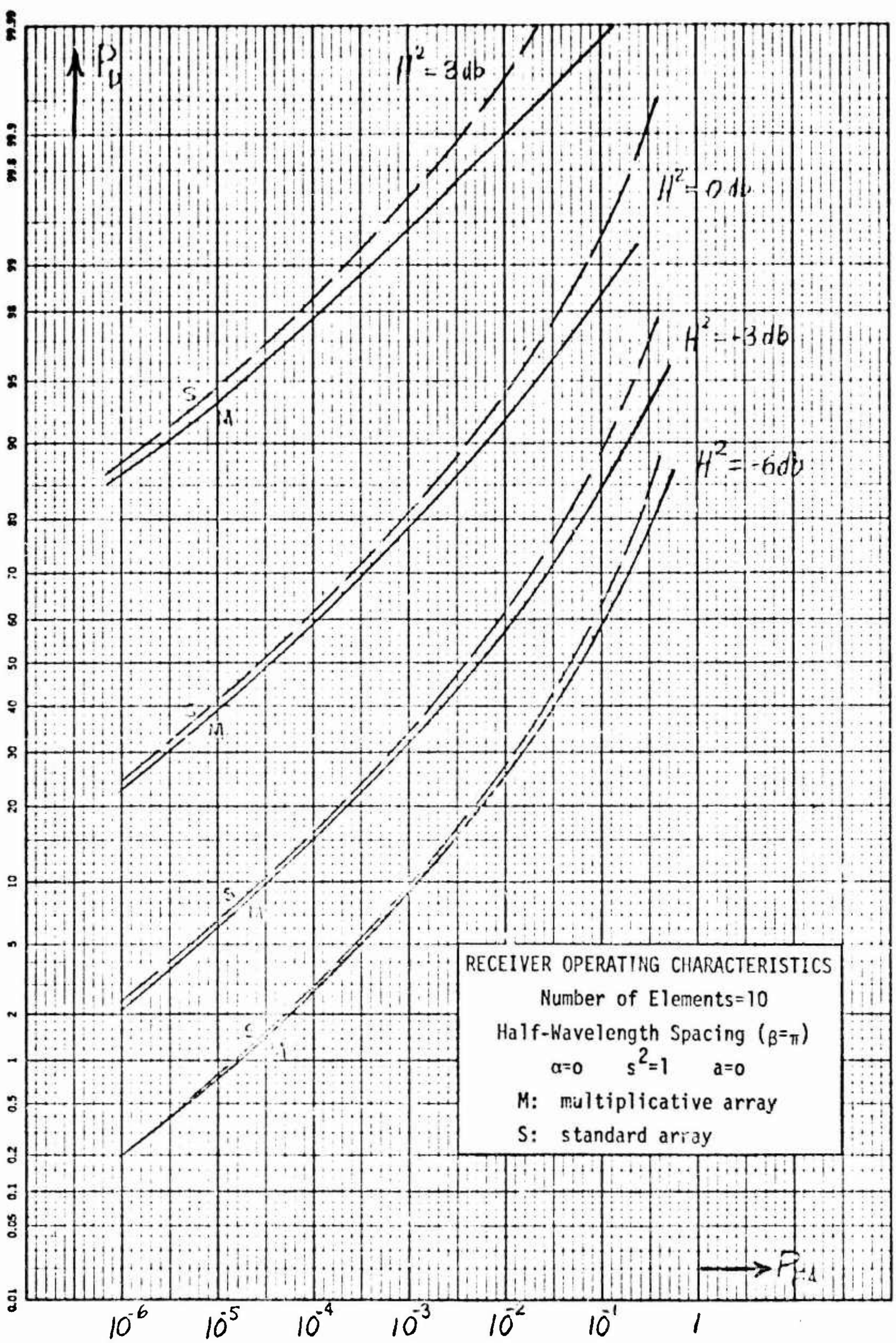


FIGURE 3-6 ROC FOR 10-ELEMENT ARRAYS

are consistently above those of the multiplicative indicates that the standard array model is a better detector in this case. They are quite close together, however, and the minimum signals for the two configurations would appear not to differ by more than 1 dB.

The more usual presentation of detection probability as a function of SNR is given in Figure 3-7 for four-element arrays and in Figure 3-8 for ten-element arrays, for various values of false alarm probability. These curves, of course, contain the same information as Figures 3-5 and 3-6, and are given here for the convenience of those who are more familiar with this presentation.

Arndt[58] compared SNR and angular resolution (beamwidth) properties of various array configuration, including linear arrays of the type we are studying. Assuming that probability of detection is proportional to output SNR, he is willing to declare that additive (conventional or "standard") processing is the better detection strategy, although for the price of 3 dB in SNR, split-beam multiplicative processing obtains superior bearing "estimation" (tracking) capability. Arndt concludes that this "price" is too high where detection is paramount, although he does not consider the false alarm rates of the two configurations.

Our results, while not directly comparable, suggest that more thorough examination may reveal that the cost in detection probability (vs. false alarm rate) is less than indicated by the difference in SNR.

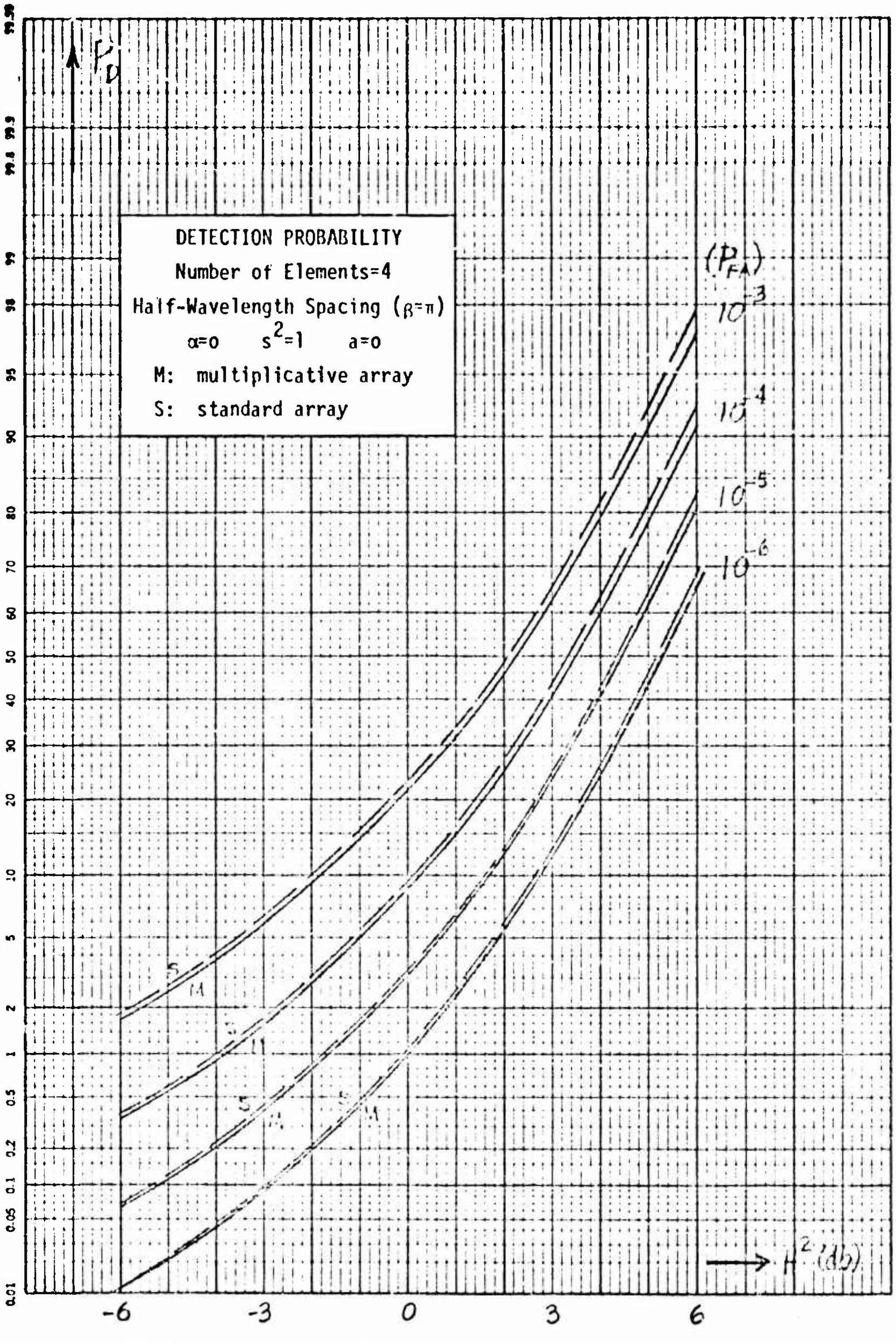


FIGURE 3-7 DETECTION PROBABILITY vs INPUT SNR (4 ELEMENTS)

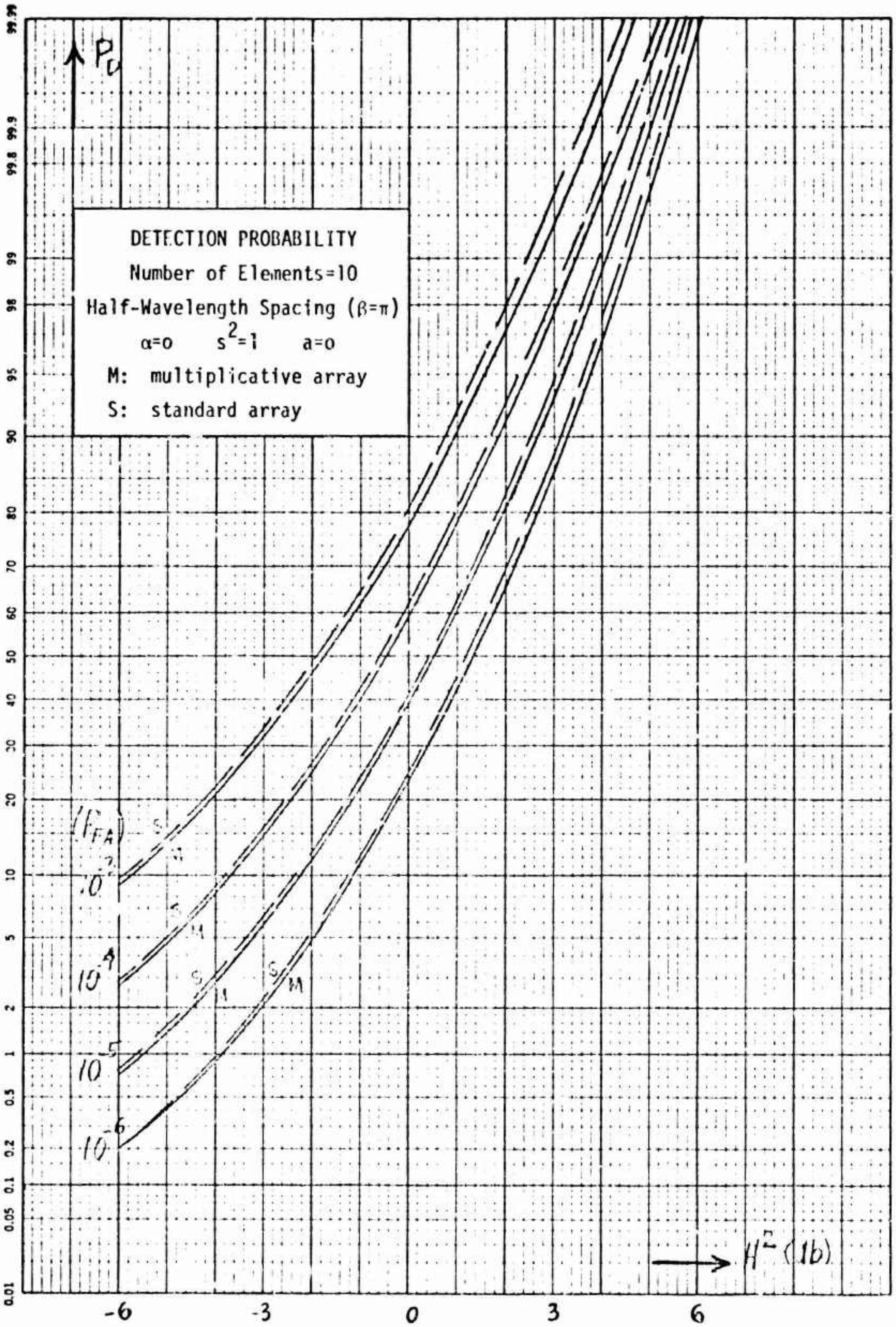


FIGURE 3-8 DETECTION PROBABILITY vs INPUT SNR
(10 ELEMENTS)

CHAPTER FOUR

MAXIMUM LIKELIHOOD BEARING ESTIMATION

In Chapter 3 we have generated characterizations of signal detection performance based on the probability density functions of Chapter 2. At this stage, we have provided the analytical equipment to answer a good many of the questions raised in the literature regarding the relative merits of the two array configurations we are considering. However, before making a general comparison, we should like to include another important consideration.

In addition to detecting the presence or absence of a signal, we desire to know the bearing of the signal source precisely. The purpose both of steering the array and of seeking to improve its directivity by summing the outputs of several hydrophones, is to obtain a filter output whose value can be depended upon to show a peak when the array is steered in the direction of the signal source. The precision of this technique is, of course, related to the tracking (noise) conditions and also to such factors as the accuracy of the steering and peak-detecting operations. (Often the "peak-detector" is a sonar operator.)

The precision with which bearing is determined can be calculated in some fashion for a given realization of an array system. However, what we are interested in discovering here is, given the array processor configuration, what is its ultimate capability in establishing bearing? An answer to this question could, for example, motivate a designer's decision to improve the performance of an array subsystem or component, if the configuration's ultimate capabilities were not nearly being realized.

In Chapter 5 we show calculations, based on statistical concepts, which bound the minimum bearing error attainable. In this chapter we derive forms for bearing estimators, whose performance we shall later compare with the theoretical bounds.

4.1 MAXIMUM LIKELIHOOD ESTIMATION

From (1-4) and (1-5), we know that the array filter output, if there is no noise, behaves in a known way as a function of the bearing, so that we can write $y(t) = y(t; \alpha)$. Assuming we have only this filter output to work with, we could determine the bearing by inverting this function: $\alpha = \alpha(y)$. However, since in general there is noise present, the result of this operation would have to be considered an estimate of the bearing, written α^*, and illustrated in Figure 4-1. And there would be error associated with the estimate. Much attention is given in statistics to the problem of finding estimators which yield the most accurate (least mean square error or minimum variance) value of the parameter being measured. Since we have the probability density function at the filter output, we may make use of the results of the statistical theory of estimation.

According to Cramer[28], the most important general method of finding estimates is the maximum likelihood method. Briefly stated, this method, as applied to our problem, consists of considering the maximum likelihood bearing estimate ($\hat{\alpha}$) to be that bearing for which the known value of the filter output (y) is the most likely value. That is,

$$\hat{\alpha} \triangleq \left\{ \hat{\alpha}:\ p(y;\ \hat{\alpha}) \geq p(y;\ \alpha) \right\}. \tag{4-1}$$

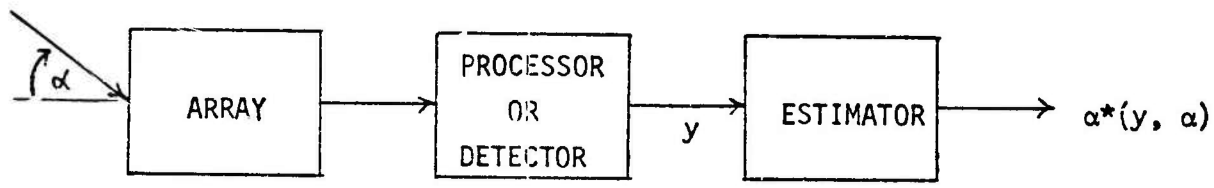


FIGURE 4-1 BEARING ESTIMATOR BASED ON FILTER OUTPUT

Therefore, we may determine the maximum likelihood estimate by finding the bearing which maximizes p(y: α):

$$\hat{\alpha} \triangleq \left\{ \hat{\alpha}(y) : \frac{\partial p(y;\alpha)}{\partial \alpha} \bigg|^{\alpha=\hat{\alpha}} = 0, \; \frac{\partial^2 p(y;\alpha)}{\partial \alpha^2} \bigg|^{\alpha=\hat{\alpha}} < 0 \right\}. \tag{4-2}$$

The importance of this method lies in the property that [29] under fairly general assumptions, the maximum likelihood method will produce asymptotically efficient estimates, that is estimates whose variance approaches the theoretical minimum for certain conditions.

We shall not be computing the second derivative indicated in (4-2) since we have seen in Chapter 2 that the bearing (suitably restricted in value) acts somewhat like a location parameter, and thus we expect that the solution to the maximum likelihood equation $\partial p/\partial\alpha = 0$ to also be the value of α which causes the given value of y to correspond to the peak of the distribution. Although for small SNR this location influence of α changes in that the peak value remains at y = 0, we shall assume that the solution of the maximum likelihood equation continues to be a maximum.

4.2 THE MAXIMUM LIKELIHOOD BEARING ESTIMATE FOR THE STANDARD ARRAY

Using (2-25), the probability density function of the filter output of the standard or square law array, we have the following equation for the maximum likelihood bearing estimate:

$$\frac{\partial p_{sq}(y)}{\partial \alpha} \bigg|^{\alpha=\hat{\alpha}} = \frac{1}{\sigma^2} \exp\left(\frac{-y}{\sigma^2}\right) \frac{\partial}{\partial \alpha}\left[e^{-h^2} I_0\left(\frac{2h}{\sigma}\sqrt{y}\right)\right] \bigg|^{\alpha=\hat{\alpha}} = 0 \tag{4-3}$$

or $$2h\, I_0\left(\frac{2h}{\sigma}\sqrt{y}\right) \frac{\partial h}{\partial \alpha}\bigg|^{\alpha=\hat{\alpha}} = \frac{2}{\sigma}\sqrt{y}\, I_1\left(\frac{2h}{\sigma}\sqrt{y}\right)\frac{\partial h}{\partial \alpha}\bigg|^{\alpha=\hat{\alpha}} \tag{4-4}$$

or $$\frac{I_1\left(\frac{2\hat{h}}{\sigma}\sqrt{y}\right)}{I_0\left(\frac{2\hat{h}}{\sigma}\sqrt{y}\right)} = \frac{\sigma\hat{h}}{\sqrt{y}}, \tag{4-5}$$

where, from Section 2.2.2, we have the notation

$$\hat{h} \triangleq h(\hat{\alpha}) = \sqrt{\frac{N}{B'}}\, HD(\tfrac{1}{2}\beta\sin\hat{\alpha};\ N). \tag{4-6}$$

Equation (4-5) corresponds to a special case of the maximum likelihood equation for the estimates of the noncentrality parameter of a non-central chi-square variate, as given by Meyer[33].

4.2.1 THE HIGH SNR CASE

As shown in Figure 4-2, the ratio of Bessel functions on the left hand side of (4-5) approaches unity for large values of the argument, satisfied by large SNR (H^2). That is,

$$I_1(x)/I_0(x) \cong 1, \quad x > 3. \tag{4-7}$$

Thus for $H^2D^2y > 2.25(B')^2s^2$ or $h^2y > 2.25\sigma^2$, we obtain

$$\hat{h} = \sqrt{y}/\sigma, \tag{4-8}$$

or, from section 2.2.2,

$$y = s^2N^2H^2D^2(\tfrac{1}{2}\beta\sin\hat{\alpha};\ N)$$

$$= s^2H^2\sin^2(\tfrac{1}{2}N\beta\sin\hat{\alpha})/\sin^2(\tfrac{1}{2}\beta\sin\hat{\alpha}), \tag{4-9}$$

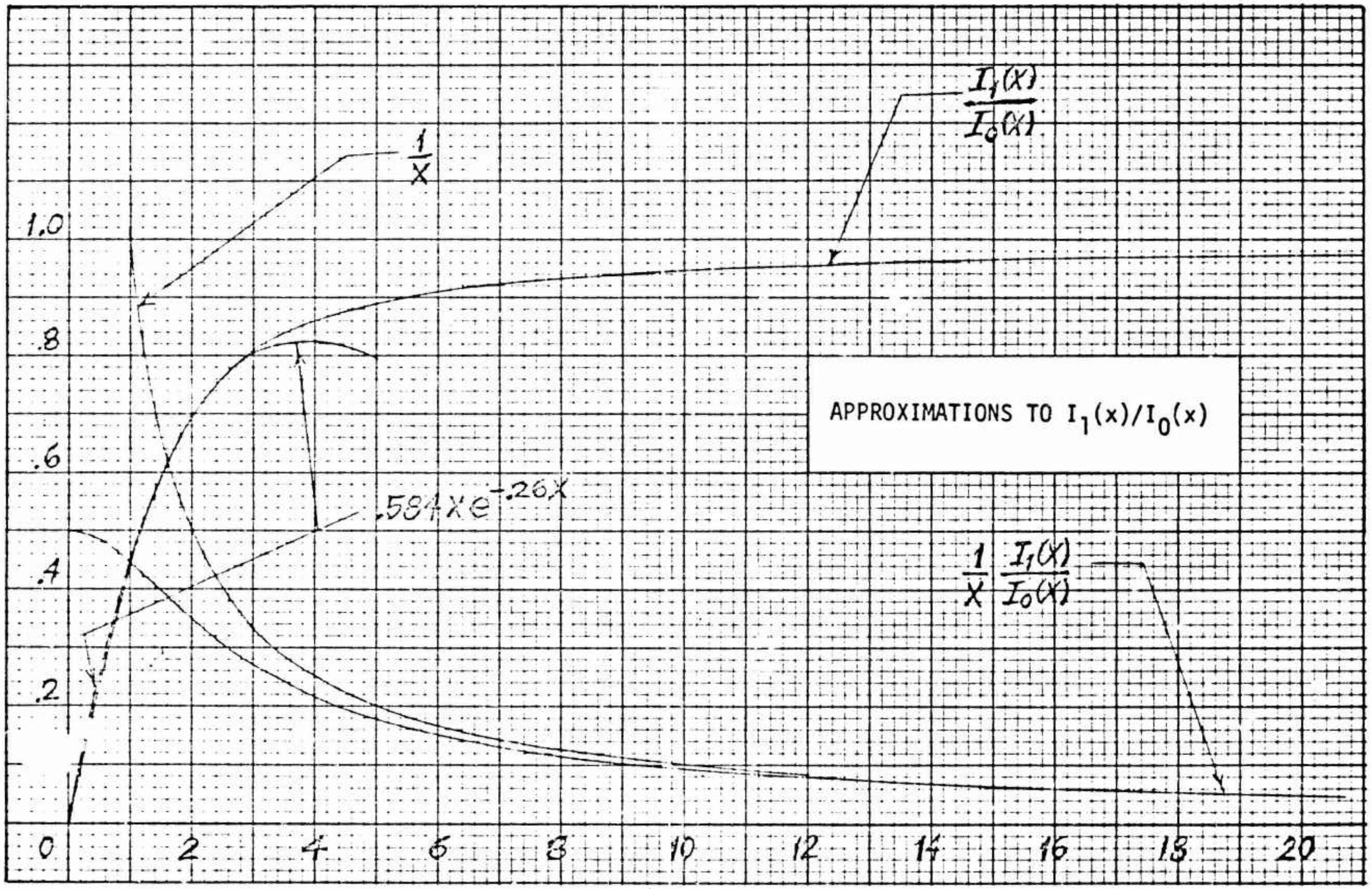

APPROXIMATIONS TO $I_1(x)/I_0(x)$
$\frac{I_1(X)}{I_0(X)}$
$\frac{1}{X}$
$.584Xe^{-.26X}$
$\frac{1}{X}\frac{I_1(X)}{I_0(X)}$
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
18
20

FIGURE 4-2

an implicit form which we shall find useful in Section 4.4. From Appendix C we have the approximate inverse or explicit forms

$$\hat{\alpha}(y) \cong \arcsin \frac{2}{\beta}\sqrt{\frac{6}{N^2-1}\left(1-\frac{\sqrt{y}}{NHs}\right)}$$

$$\cong \frac{1}{2} \arccos\left\{1 + \frac{48}{\beta^2(N^2-1)}\left(\frac{\sqrt{y}}{NHs} - 1\right)\right\}, \quad \sqrt{y} < NHs \qquad (4\text{-}10)$$

$$= 0, \quad \sqrt{y} > NHs$$

where the approximation assumes $\frac{1}{2}(N-1)\beta \sin \hat{\alpha} << \sqrt{12}$. Since the beamwidth is given at most by $|\frac{1}{2}N\beta \sin \alpha| < \pi$, this assumption is a valid one.

It is instructive to note that (4-9) is equivalent to taking

$$y(\hat{\alpha}) = [E(y) - E(y|H = 0)]^{\alpha=\hat{\alpha}}, \qquad (4\text{-}11)$$

that is, the maximum likelihood estimate for the high SNR case can be interpreted as the operation which assumes that the filter output is its mean value, less the 1 ffects of noise.

4.2.2 THE LOW SNR CASE

For small values of the argument in (4-5), the ratio of the Bessel functions can be approximated by

$$I_1(x)/I_0(X) = .584x\, e^{-.26x}, \quad x < 3. \qquad (4\text{-}12)$$

Thus for $H^2D^2y < 2.25(B')^2s^2$, we obtain

$$.584\, \frac{2}{\sigma}\sqrt{y}\, \exp \frac{-.52\hat{h}\sqrt{y}}{\sigma} = \sigma/\sqrt{y} \qquad (4\text{-}13)$$

or $$\hat{h} = \frac{\sigma}{.52\sqrt{y}} \ln\left(\frac{1.168y}{\sigma^2}\right) \tag{4-14}$$

or, from section 2.2.2,

$$D(\tfrac{1}{2}N\beta\sin\hat{\alpha};\ N) = \frac{[N+2(N-1)a]\ s}{.52NH\sqrt{y}} \ln\left(\frac{1.168y/s^2}{N+2(N-1)a}\right) . \tag{4-15}$$

As for (r-10), we use (D-9) to write

$$\hat{\alpha}(y) = \tfrac{1}{2}\arccos\left\{1 - \frac{48(1-D)}{\beta^2(N^2-1)}\right\} .$$

$$= \tfrac{1}{2}\arccos\left\{1 - \frac{48}{\beta^2(N^2-1)}\left[1 - \frac{[N+2(N-1)a]\ s}{.52NH\ y}\ln\frac{1.168y/s^2}{N+2(N-1)a}\right]\right\} \tag{4-16}$$

4.3 THE MAXIMUM LIKELIHOOD BEARING ESTIMATE FOR THE MULTIPLICATIVE ARRAY

For the multiplicative array model, the maximum likelihood equation may be written

$$\left.\frac{\partial P_6(y;\ \alpha)}{\partial\alpha}\right|^{\alpha=\hat{\alpha}} = \left.\frac{\partial P_6}{\partial\theta}\frac{d\theta}{d\alpha}\right|^{\alpha=\hat{\alpha}} = \left.\tfrac{1}{2}M\beta\cos\alpha\frac{\partial P_6}{\partial\theta}\right|^{\alpha=\hat{\alpha}} = 0. \tag{4-17}$$

Assuming that $\cos\hat{\alpha} \neq 0$, we have, using $\hat{\theta} = \theta(\hat{\alpha})$,

$$\left.\frac{\partial P_6}{\partial\theta}\right|^{\theta=\hat{\theta}} = 0. \tag{4-18}$$

From [13], formulas 8.972.1 and 9.212.1, we observe that

$$e^x L_r^k(-x) = \binom{r+k}{r} {}_1F_1(r+k+1, k+1; x). \tag{4-19}$$

Substituting this expression in (2-23), we have for the probability density of the filter output for $h_1 = h_2 = h$

$$p_6(y;\alpha) = g(\theta) = a_1 \exp[-a_2 h^2(1-\rho\cos 2\theta)] \sum_{r=0}^{\infty} a_3^r (h\cos\theta)^r$$

$$\times\; I_r(a_4 h\cos\theta)\; {}_1F_1(r+1, 1;\, a_5 h^2 \sin^2\theta), \tag{4-20}$$

where we have tried to simplify the notation somewhat by writing

$a_1 = \sigma^{-2}\exp[-2y/\sigma^2(1+\rho)]$, $a_2 = 2/(1-\rho^2)$, $a_3 = (1-\rho)\sigma/2\sqrt{y}$, $a_4 = 4\sqrt{y}/\sigma(1+\rho)$, and $a_5 = (1+\rho)/(1-\rho)$.

Performing the required differentiation of (4-20) with respect to θ, we obtain

$$\begin{aligned} g'(\theta) = {} & a_1 \exp[-a_2 h^2(1-\rho\cos 2\theta)] \\ & \times \Big\{ -2a_2 h[h'(1-\rho\cos 2\theta) + \rho h \sin 2\theta] \sum_{r=0}^{\infty} a_3^r (h\cos\theta)^r I_r(\;)\, {}_1F_1(\;) \\ & + (h'\cos\theta - h\sin\theta) \sum_{r=0}^{\infty} r a_3^r (h\cos\theta)^{r-1}\, I_r(\;)\, {}_1F_1(\;) \\ & + \tfrac{1}{2} a_4 (h'\cos\theta - h\sin\theta) \sum_{r=0}^{\infty} a_3^r (h\cos\theta)^r [I_{r+1}(\;) + I_{r-1}(\;)]\, {}_1F_1(\;) \\ & + 2a_5 h \sin\theta (h'\sin\theta + h\cos\theta) \sum_{r=0}^{\infty} (r+1) a_3^r (h\cos\theta)^r I_r(\;) \\ & \times {}_1F_1(r+2, 2;\, a_5 h^2 \sin^2\theta) \Big\} \end{aligned} \tag{4-21}$$

In (4-21) we have used $h' = dh/d\theta$ and have suppressed the arguments of functions whenever they are the same as in (4-20). By combining summations and noting that $rI_r(x) = x[I_{r-1}(x) - I_{r+1}(x)]$, we may reduce (4-21) to

$$
\begin{aligned}
g'(\theta) = {} & a_1 \exp[-a_2 h^2(1-\rho\cos 2\theta)] \sum_{r=0}^{\infty} (a_3 h\cos\theta)^r \\
& \times \Big\{ -2a_2[h'(1-\rho\cos 2\theta) + \rho h\sin 2\theta] I_r(a_4 h\cos\theta)\, {}_1F_1(r+1,1;a_5 h^2\sin^2\theta) \\
& + a_4(h'\cos\theta - h\sin\theta) I_{r-1}(a_4 h\cos\theta)\, {}_1F_1(r+1,1;a_5 h^2\sin^2\theta) \\
& + 2a_5(r+1) h\sin\theta(h'\sin\theta + h\cos\theta) I_r(a_4 h\cos\theta)\, {}_1F_1(r+2,2;a_5 h^2\sin^2\theta) \Big\}.
\end{aligned}
\tag{4-22}
$$

A slightly more compact form may be realized by using (4-19) to return to the Laguerre polynomials:

$$
\begin{aligned}
g'(\theta) = {} & a_1 \exp[-a_2 h^2(1-\rho\cos 2\theta) + a_5 h^2\sin^2\theta] \sum_{r=0}^{\infty} (a_3 h\cos\theta)^r \\
& \times \Big\{ -2a_2[h'(1-\rho\cos 2\theta) + \rho h\sin 2\theta] I_r(a_4 h\cos\theta) L_r(-a_5 h^2\sin^2\theta) \\
& + a_4(h'\cos\theta - h\sin\theta) I_{r-1}(a_4 h\cos\theta) L_r(-a_5 h^2\sin^2\theta) \\
& + 2a_5 h\sin\theta(h\sin\theta + h\cos\theta) I_r(a_4 h\cos\theta) L_r^1(-a_5 h^2\sin^2\theta) \Big\}.
\end{aligned}
\tag{4-23}
$$

Our task is to solve the equation $g'(\hat{\theta}) = 0$, for $\hat{\theta} = \hat{\theta}(y)$. However, this we have not been able to do in general. Rather, we next show a small SNR approximation to the solution, and for higher SNR, a different approach.

4.3.1 SMALL SNR CASE

For h sufficiently small, we may find an approximate solution by taking the first term of the series. Thus $g'(\theta) = 0$ becomes

$$0 = -2a_2[h'(1-\rho\cos 2\theta)+\rho h\sin 2\theta]I_0(a_4 h\cos\theta)$$

$$+ a_4(h'\cos\theta - h\sin\theta)I_1(a_4 h\cos\theta)$$

$$+ 2a_5 h\sin\theta(h'\sin\theta + h\cos\theta)I_1(a_4 h\cos\theta). \qquad (4\text{-}24)$$

Dropping the hh' terms and supplying the expressions represented by a_2 and a_4, we obtain

$$\frac{1}{1-\rho^2}[h'(1-\rho\cos 2\theta)+\rho h\sin 2\theta] = \frac{4\sqrt{y}}{\sigma(1+\rho)}(h'\cos\theta - h\sin\theta)\frac{I_1[4h\cos\theta\sqrt{y}/\sigma(1+\rho)]}{I_0[4h\cos\theta\sqrt{y}/\sigma(1+\rho)]}. \qquad (4\text{-}25)$$

For convenience we use a different approximation to the ratio of the Bessel functions than (4-12). From Figure 4-2 we see that if x is quite small, then

$$\frac{I_1(x)}{I_0(x)} \cong \frac{x}{2}, \; x < 1. \qquad (4\text{-}26)$$

With this approximation, (4-25) can be manipulated to yield the implicit estimator form,

$$y = \frac{\sigma^2}{h\cos\hat{\theta}}\left(\frac{1+\rho}{1-\rho}\right)\frac{\hat{h}'(1-\rho\cos 2\hat{\theta})+\hat{\rho}h\sin 2\hat{\theta}}{\hat{h}'\cos\hat{\theta}-\hat{h}\sin\hat{\theta}}. \qquad (4\text{-}27)$$

In the terms of Section 2.2.1,

$$y = \frac{s^2 B^{3/2}}{H D(\hat{\theta}/M;M)} \left(\frac{B+C}{B-C}\right) \frac{D'(\hat{\theta}/M;M)(B-C\cos 2\hat{\theta})+CD(\hat{\theta}/M;M)\sin 2\hat{\theta}}{D'(\hat{\theta}/M;M)\cos\hat{\theta}-D(\hat{\theta}/M;M)\sin\hat{\theta}}, \tag{4-28}$$

$$\text{where } D'(\theta/M;M) = \frac{dD}{d\theta} = \frac{1}{M\sin(\theta/M)}\left[\cos\theta - \frac{\sin\theta \cos(\theta/M)}{M\sin(\theta/M)}\right]$$

$$= \frac{D}{\sin\theta}\left[\cos\theta - D\cos(\theta/M)\right]. \tag{4-29}$$

For white noise (B = 1 and C = 0), the implicit estimator form (4-28) becomes

$$y = \frac{s^2\hat{D}'}{H\hat{D}(\hat{D}'\cos\hat{\theta}-\hat{D}\sin\hat{\theta})} = \frac{s^2}{H\hat{D}}\,\frac{\cos\hat{\theta}-\hat{D}\cos(\hat{\theta}/M)}{\cos 2\hat{\theta}-\hat{D}\cos\hat{\theta}\cos(\hat{\theta}/M)}. \tag{4-30}$$

For small θ (within the array major lobe), we may approximate:

$$y = \frac{s^2}{H\hat{D}}\,\frac{1-\hat{\theta}^2/2-\hat{D}(1-\hat{\theta}^2/2M^2)}{1-2\hat{\theta}^2-\hat{D}(1-\hat{\theta}^2/2)(1-\hat{\theta}^2/2M^2)} \cong \frac{s^2}{H\hat{D}}\,\frac{1-\hat{D}-\hat{\theta}^2/2}{1-\hat{D}-3\hat{\theta}^2/2} \tag{4-31}$$

$$\text{or} \qquad y \cong s^2/3H\hat{D}. \tag{4-32}$$

From this we obtain the explicit estimator form for the white noise, small SNR case,

$$\hat{\alpha}(y) = \tfrac{1}{2}\arccos\left[1 - \frac{48}{\beta^2(M^2-1)}\left(1 - \frac{s^2}{3Hy}\right)\right] \tag{4-33}$$

4.3.2 A NONCENTRAL CHI-SQUARE APPROXIMATION

For SNR other than very small, rather than work with (4-23) we shall follow the procedure of approximating the density function of the filter output y by one which has a closed form, and then deriving the maximum likelihood bearing estimate for the approximate distribution.

Over a wide range of values for SNR, we have found that a good approximation to the probability density function (2-23) for the multiplicative array model is the density function of an equivalent noncentral chi-square random variable, chosen such that the mean and variance of the approximate distribution are the same as the original.

Let $x = d + cy/\sigma^2$ be a noncentral chi-square variate with two degrees of freedom and noncentrality parameter λ^2. Then we have

$$p_y(y) = cp_x(cy/\sigma^2+d)/\sigma^2 = \frac{c}{2\sigma^2}\exp[-(cy/\sigma^2+d+\lambda^2)/2]$$

$$\times\, I_0(\lambda\sqrt{cy/\sigma^2+d}),\quad cy/\sigma^2+d > 0. \tag{4-34}$$

From Section 2.3.1,

$$E(y) = \sigma^2(\rho+h^2\cos 2\theta) = \frac{\sigma^2}{c}[E(x)-d] = \frac{\sigma^2}{c}(2+\lambda^2-d) \tag{4-35}$$

$$\mathrm{Var}(y) = \frac{\sigma^4}{2}[1+\rho^2+2h^2(1+\cos 2\theta)] = \frac{\sigma^4}{2}\mathrm{Var}(x) = \frac{4\sigma^4(1+\lambda^2)}{c^2}. \tag{4-36}$$

In terms of c we can solve (4-35) and (4-36) to yield

$$\lambda^2 = \frac{c^2}{8}[1+\rho^2+2h^2(1+\rho\cos 2\theta)] - 1 \tag{4-37}$$

$$d = 1+\frac{c^2}{8}[1+\rho^2+2h^2(1+\rho\cos 2\theta)] - c(\rho+h^2\cos 2\theta). \tag{4-38}$$

An illustration of this approximation is given by Figure 4-3.

Again assuming $\cos\alpha \neq 0$, (4-18) becomes

$$\left.\frac{\partial p_y(y)}{\partial\theta}\right|^{\theta=\hat{\theta}} = 0. \tag{4-39}$$

For $p_y(y)$ as given in (4-34), the maximum likelihood equation then is

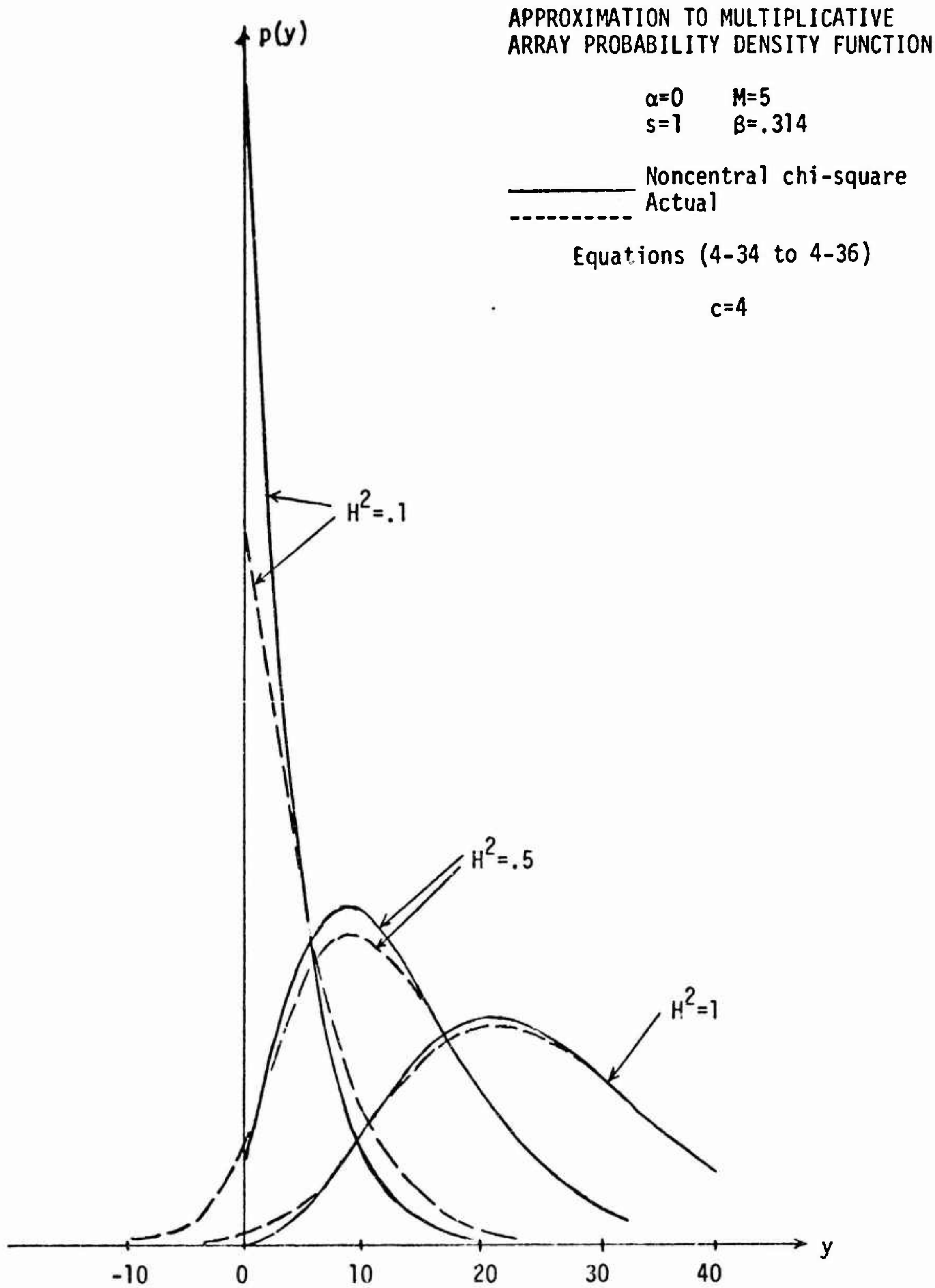

p(y)
APPROXIMATION TO MULTIPLICATIVE
ARRAY PROBABILITY DENSITY FUNCTION
α=0 M=5
s=1 β=.314
Noncentral chi-square
Actual
Equations (4-34 to 4-36)
c=4
H²=.1
H²=.5
H²=1
-10
0
10
20
30
40
y

FIGURE 4-3

$$(\hat{d}'+2\hat{\lambda}\hat{\lambda}')I_0(\hat{\lambda}\sqrt{cy/\sigma^2+\hat{d}})$$

$$= I_1(\hat{\lambda}\sqrt{cy/\sigma^2+\hat{d}})\ [2\hat{\lambda}'\sqrt{cy/\sigma^2+\hat{d}} + \hat{\lambda}\hat{d}'/\sqrt{cy/\sigma^2+\hat{d}}\ ] \tag{4-40}$$

or, using $x = cy/\sigma^2+\hat{d}$,

$$\frac{I_1(\hat{\lambda}\sqrt{x})}{I_0(\hat{\lambda}\sqrt{x})} = \frac{\hat{d}' + 2\hat{\lambda}\hat{\lambda}'}{2\lambda'\sqrt{x} + \lambda d'/\sqrt{x}} \ . \tag{4-41}$$

Taking the ratio of the Bessel functions to be unity, we obtain the quadratic equation,

$$2\hat{\lambda}'x - (\hat{d}'+2\hat{\lambda}\hat{\lambda}')\sqrt{x} + \hat{\lambda}\hat{d}' = 0, \tag{4-42}$$

with the solutions

$$\sqrt{x} = \hat{d}'/2\hat{\lambda}',\hat{\lambda} \text{ or } cy/\sigma^2 + \hat{d} = (\hat{d}'/2\hat{\lambda}')^2,\ \hat{\lambda}^2. \tag{4-43}$$

Putting c = 4 into (4-37) and (4-38), the first solution can be expressed

$$\frac{y_1}{\sigma^2} = \frac{1}{16}\left[\frac{\hat{h}'[1-(1-\rho)\cos 2\hat{\theta}] + \hat{h}(1-\rho)\sin 2\hat{\theta}}{\hat{h}'(1+\rho\cos 2\hat{\theta}) - \hat{h}\rho\sin 2\hat{\theta}}\right]^2 - 3 + 2\rho(2-\rho)$$

$$+ 4\hat{h}^2[1-(1-\rho)\cos 2\hat{\theta}]. \tag{4-44}$$

Note the slight resemblance of this solution to (4-27). The second solution has the much simpler form

$$y_2 = \sigma^2\hat{h}^2\cos 2\hat{\theta} - \sigma^2(\tfrac{1}{2}-\rho)$$

$$= Ms^2[MH^2\hat{D}^2\cos 2\hat{\theta}+C-B/2]. \tag{4-45}$$

This solution also has the form, or nearly so, of $E(y) - E(y|H=0)$. For this reason, and for its simplicity, we shall restrict our attention to it. An additional motive for preferring (4-45) is that, as the SNR becomes very large and $p_6(y)$ approaches the normal distribution, (4-45) corresponds to the well known maximum likelihood estimate for that case [57].

To obtain an explicit form $\hat{\alpha}(y)$ for the estimate, we first note that (4-45) can be written

$$D^2(\hat{\theta}/M;\ M)\cos 2\hat{\theta} = \frac{\sin^2\hat{\theta}\,\cos 2\hat{\theta}}{M^2\sin^2(\hat{\theta}/M)} = \frac{1}{MH^2}\left(\frac{y}{Ms^2} + \frac{B}{2} - C\right) \tag{4-46}$$

$$= \frac{\sin^2 2\hat{\theta}}{4M^2\sin^2(\hat{\theta}/M)} - \frac{\sin^4\hat{\theta}}{M^2\sin^2(\hat{\theta}/M)}$$

$$= D^2(\hat{\theta}/M;\ 2M) - M^2\sin^2(\hat{\theta}/M)D^4(\hat{\theta}/M;\ M)$$

$$\approx D^2(\hat{\theta}/M;\ 2M). \tag{4-47}$$

Then, using (D-9) and recalling that $\theta = \frac{1}{2}M\beta\sin\alpha$, we find

$$\hat{\alpha}(y) = \tfrac{1}{2}\arccos\left[1 - \frac{48}{\beta^2(4M^2-1)}\sqrt{\frac{1}{MH^2}\left(\frac{y}{Ms^2} + \frac{B}{2} - C\right)}\right]. \tag{4-48}$$

4.4 DISTRIBUTIONS OF THE ESTIMATES

The estimate $\hat{\alpha}(y)$, being a transformation of a random variable, is itself a random variable, whose probability density function is given by

$$p_A(\hat{\alpha};\ \alpha) = p_y[y(\hat{\alpha});\ \alpha]\ |dy/d\hat{\alpha}|. \tag{4-49}$$

For example, for the multiplicative array, high SNR case, the estimate (4-45) has the density function

$$p_A(\hat{\alpha}; \alpha) = p_6[\sigma^2\hat{h}^2\cos 2\hat{\theta} - \sigma^2(\tfrac{1}{2}-\rho);\ \alpha]2\sigma^2\hat{h}|\hat{h}'\cos 2\hat{\theta} - \hat{h}\sin 2\hat{\theta}|\hat{\theta}', \quad (4\text{-}50)$$

where we continue to use the shorthand notations $\hat{h} = h(\hat{\theta})$, $\hat{\theta} = \theta(\hat{\alpha})$.

Often at this point the analyst will attempt to provide an expression for $p_A(\hat{\alpha};\alpha)$ in the form $p_A(\hat{\alpha};\alpha) = f(\hat{\alpha}-\alpha)$. However, since the expressions for the density functions are so complex, this objective requires much approximation, and there is little to be gained. We shall be satisfied here to note that these densities are readily computed from (4-49). Instead, this effort will be put into showing the means and variances of the estimates for the high SNR cases so that their asumptotic behavior may be examined in relationship to the theoretical bounds in the next chapter.

4.4.1 MEAN AND MEAN SQUARE, THE STANDARD ARRAY ESTIMATE

The high SNR bearing estimate for the standard array, given by (4-10), has the mean value

$$E(\hat{\alpha}) = \sigma^{-2}\int_0^{N^2H^2s^2} dy\ \arcsin[K\sqrt{1-\sqrt{y}/NHs}]\exp(-h^2-y/\sigma^2)I_0(2h\sqrt{y}/\sigma)$$

$$= k\int_0^1 dx\ \arcsin[K\sqrt{1-\sqrt{x}}]\exp(-h^2-kx)I_0(2h\sqrt{kx})$$

$$= 2k\int_0^1 dx\ x\ \arcsin[K\sqrt{1-x}]\exp(-h^2-kx^2)I_0(2hx\sqrt{k}), \quad (4\text{-}51)$$

where we employ $K = \frac{2}{\beta}\sqrt{\frac{6}{N^2-1}}$ and $k = \frac{N^2H^2s^2}{\sigma^2} = h^2/D^2(\frac{1}{2}\beta\sin\alpha;\ N)$. Approximately we have

$$E(\hat{\alpha}) = 2kKe^{-h^2}\int_0^1 dx\ e^{-kx^2}\ x\sqrt{1-x}\ I_0(2hx\sqrt{k})$$

$$= 2kKe^{-h^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(kh^2)^n}{n!n!}\ \frac{(-k)^m}{m!}\int_0^1 dx\ x^{2n+2m+1}\sqrt{1-x}$$

$$= 2kKe^{-h^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(kh^2)^n}{n!n!}\ \frac{(-k)^m}{m!}\ B(3/2,\ 2n+2m+2), \tag{4-52}$$

where B is the beta function, $B(u,v) = \Gamma(u)\ \Gamma(v)/\Gamma(u+v)$. By a simple change of summation index we may write

$$E(\hat{\alpha}) = 2kKe^{-h^2}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(kh^2)^n}{n!n!}\ \frac{(-k)^{m-n}}{(m-n)!}\ B(3/2,\ 2m+2)$$

$$= 2kKe^{-h^2}\sum_{m=0}^{\infty}\frac{(-k)^m}{m!}\ B(3/2,\ 2m+2)\ L_m(h^2). \tag{4-53}$$

By a similar process we find that

$$E[(\hat{\alpha})^2] = 2kK^2e^{-h^2}\sum_{m=0}^{\infty}\frac{(-k)^m}{m!}\ B(2,\ 2m+2)\ L_m(h^2). \tag{4-54}$$

The beta functions $B(3/2,\ 2n+2m+2)$, as shown in Figure 4-4, can be approximated by the geometric progression

$$B(3/2,\ 2n+2m+2) = B(3/2,2)x(.415)^{n+m}\ . \tag{4-55}$$

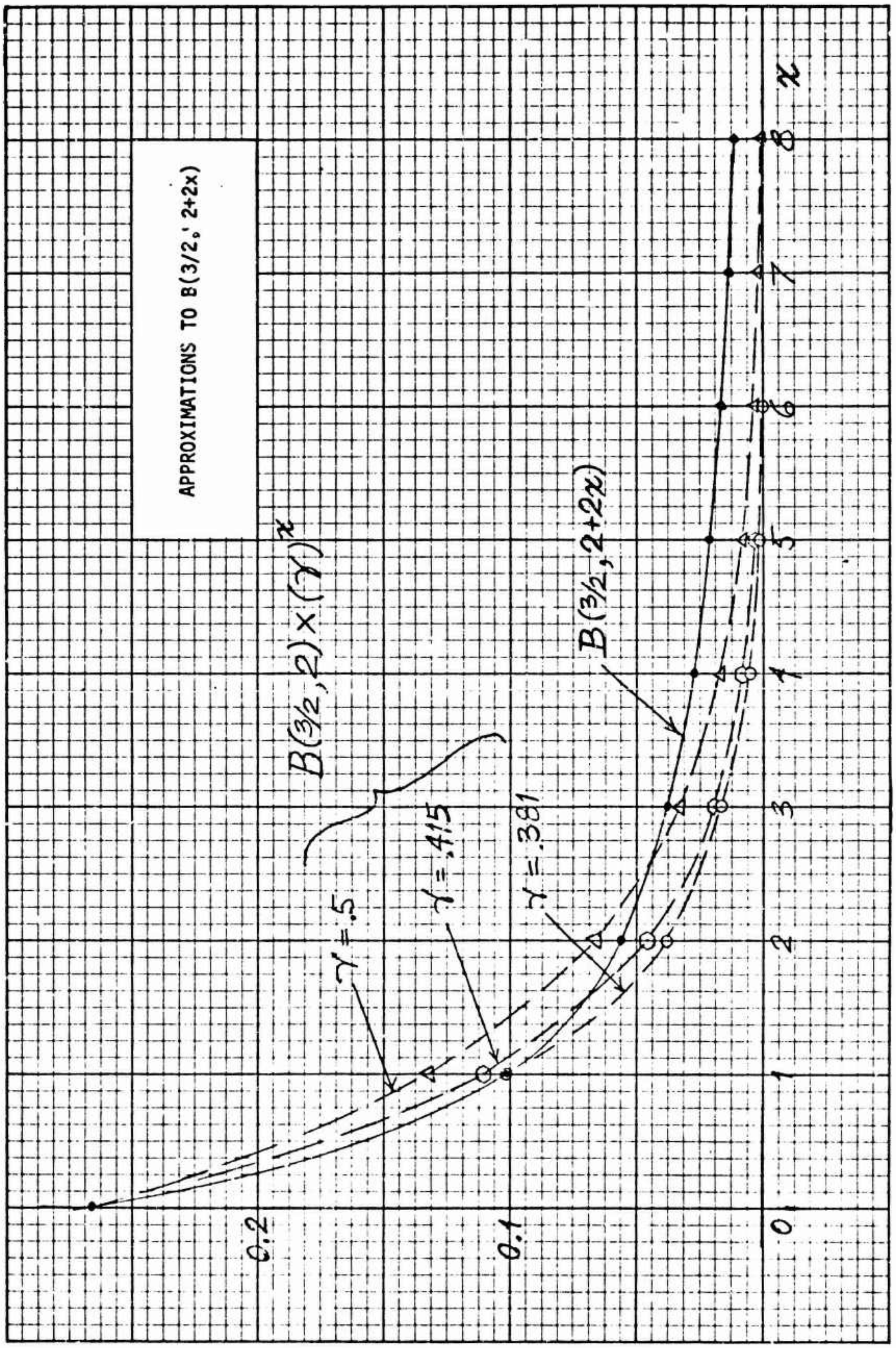

APPROXIMATIONS TO B(3/2, 2+2x)
$B(3/2,2) \times (\gamma)^x$
$B(3/2, 2+2x)$
$\gamma = .5$
$\gamma = .415$
$\gamma = .381$
0.2
0.1
0
1
2
3
4
5
6
7
8
x

FIGURE 4-4

Substituting this approximation in (4-52), we obtain

$$E(\hat{\alpha}) = 2kKe^{-h^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(.415kh^2)^n}{n!n!} \frac{(-.415k)^m}{m!} B(3/2, 2)$$

$$= 2kKB(3/2, 2)\exp(-h^2-.415k)\ I_0(2h\sqrt{.415k}) \tag{4-56}$$

$$\text{or}\quad E(\hat{\alpha}) = kKB(3/2, 2)\exp[-(h-\sqrt{.415k})^2]/\sqrt{\pi\sqrt{.415k}h}, \tag{4-57}$$

where we have used the large argument relation for the Bessel function, $I_0(x) = e^x/\sqrt{2\pi x}$. Substituting for K, k, and h, we find

$$E(\hat{\alpha}) = \frac{.916}{\beta\sqrt{D(N^2-1)}} \left(\frac{NH^2}{B'}\right)^{1/2} \exp[-NH^2(D-.644)^2/B']. \tag{4-58}$$

In a similar manner we approximate $E[(\hat{\alpha})^2]$ by using

$$B(2, 2n+2m+2) = B(2, 2)x(.335)^{n+m}. \tag{4-59}$$

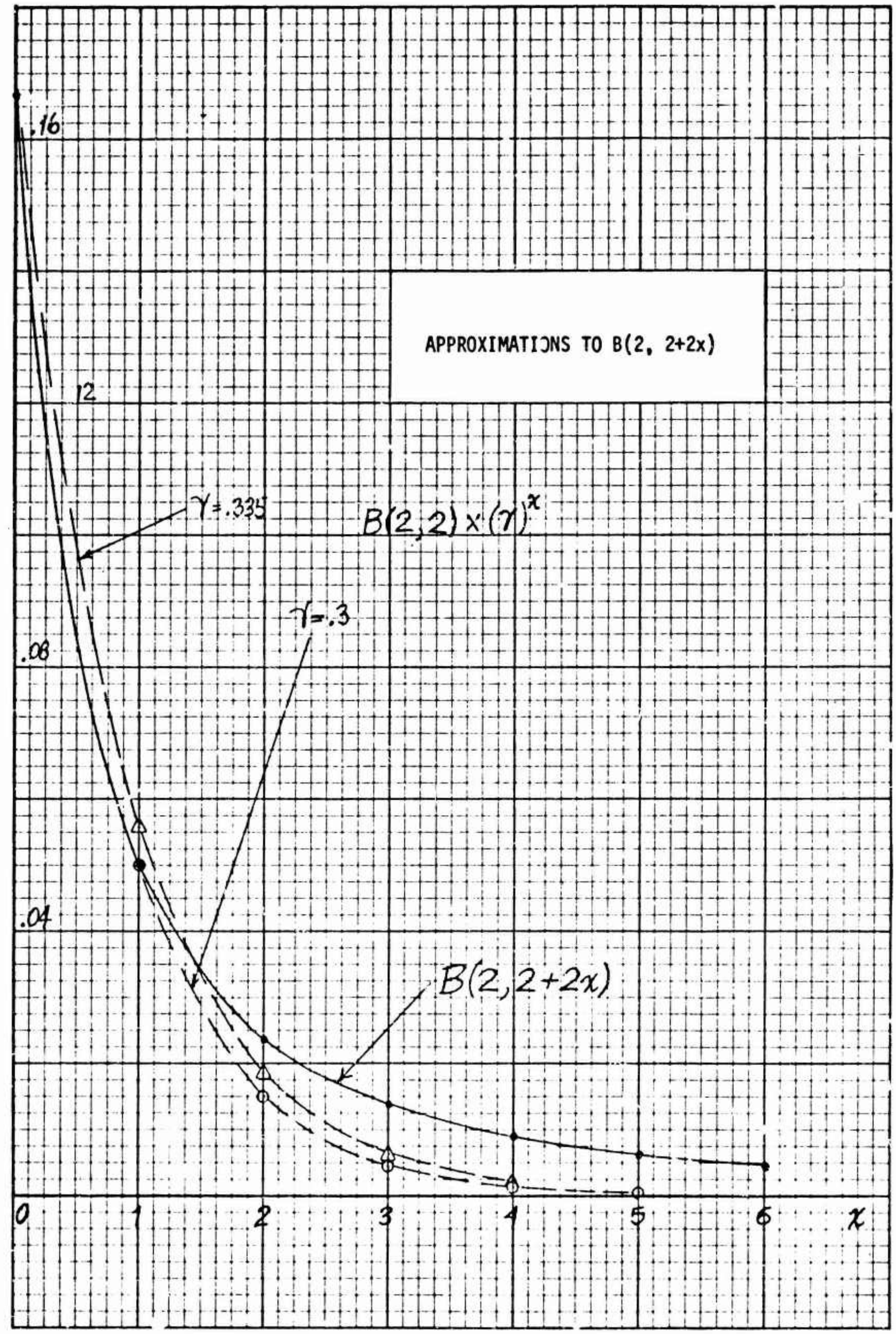
This choice of approximation is illustrated in Figure 4-5. From (4-54), the mean square of the bearing estimate becomes

$$E[(\hat{\alpha})^2] = 2kK^2e^{-h^2} \sum_{m=0}^{\infty} \frac{(-.335k)^m}{m!} B(2, 2)\ L_m(h^2)$$

$$= 2kK^2B(2, 2)\exp(-h^2-.335k)\ I_0(2h\sqrt{.335k}) \tag{4-60}$$

$$\text{or}\quad E[(\hat{\alpha})^2] = kK^2B(2, 2)\exp[-(h-\sqrt{.335k})^2]/\sqrt{\pi\sqrt{.335k}h}\ . \tag{4-61}$$

Substituting for K, k, and h, the expression for the mean square is



.16
.08
.04
APPROXIMATIONS TO B(2, 2+2x)
12
γ=.335
B(2,2)×(γ)^x
γ=.3
B(2,2+2x)
0
1
2
3
4
5
6
x

FIGURE 4-5

$$E[(\hat{\alpha})^2] = \frac{2.97}{\beta^2\sqrt{D}(N^2-1)}\left(\frac{NH^2}{B'}\right)^{1/2}\exp[-NH^2(D-.579)^2/B']. \quad (4\text{-}62)$$

These approximations (4-62) and (4-58) are useful only for smaller values of SNR ($k = NH^2/B' < 1$).

4.4.2 MEAN AND MEAN SQUARE, THE MULTIPLICATIVE ARRAY ESTIMATE

For the multiplicative array model, the high SNR bearing estimate we have developed is given by (4-48), which can also be written

$$\begin{aligned}
\hat{\alpha}(y) &= \arcsin K[1-\sqrt{(y/\sigma^2+\tfrac{1}{2}-\rho)/k}]^{1/2}, \quad 0 < y/\sigma^2+\tfrac{1}{2}-\rho < k \\
&= \arcsin K, \quad y/\sigma^2+\tfrac{1}{2}-\rho < 0 \qquad (4\text{-}63) \\
&= 0, \quad y/\sigma^2+\tfrac{1}{2}-\rho > k,
\end{aligned}$$

where, in analogy with the last section we here use $K = \frac{2}{\beta}\sqrt{\frac{6}{4M^2-1}}$ and $k = MH^2/B = h^2/D^2(\tfrac{1}{2}\beta\sin\alpha;\ M)$.

To obtain the mean and mean square of $\hat{\alpha}(y)$, we shall use the noncentral chi-square approximate density (4-34), with the constant c chosen to be 4. The mean of the estimate is then

$$\begin{aligned}
E(\hat{\alpha}) = &\int_{-\sigma^2 d/4}^{\sigma^2(\rho-\frac{1}{2})} dy \arcsin K\, p_y(y) \\
&+ \int_{\sigma^2(\rho-\frac{1}{2})}^{\sigma^2(k+\rho-\frac{1}{2})} dy \arcsin\ K[1-\sqrt{(y/\sigma^2+\tfrac{1}{2}-\rho)/k}]^{1/2}\, p_y(y).
\end{aligned} \qquad (4\text{-}64)$$

The first integral is approximately

$$\int_0^{4(\rho-\frac{1}{2})+d} dy\, K(\sigma^2/4)\, p_y[\sigma^2(x-d)/4] = \frac{K}{2} \int_0^{4(\rho-\frac{1}{2})+d} dx\, \exp[-(x+\lambda^2)/2]\, I_0(\lambda\sqrt{x})$$

$$= K - KQ\left[\lambda, \sqrt{4(\rho-\tfrac{1}{2})+d}\right], \tag{4-65}$$

Q being Marcum's Q-function.

The second integral, making the same approximation as for the first, is

$$K \int_0^1 dx\, k\sigma^2\sqrt{1-\sqrt{x}}\, p_y[\sigma^2(kx+\rho-\tfrac{1}{2})]$$

$$= 2Kk \int_0^1 dx\sqrt{1-\sqrt{x}}\, \exp[-(4kx+4\rho-2+d+\lambda^2)/2]\, I_0(\lambda\sqrt{4kx+4\rho-2+d}) \tag{4-66}$$

$$= 4Kk\, \exp(-\lambda^2/2-2g) \int_0^1 dx\, x\sqrt{1-x}\, e^{-2kx^2}\, I_0[2\lambda\sqrt{kx^2+g}]$$

$$= 4Kk\, \exp(-\lambda^2/2-2g) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(g\lambda^2)^n}{n!n!}\, \frac{(-2k)^m}{m!}$$

$$\times \int_0^1 dx\, x^{2m+1}\sqrt{1-x}(1+kx^2/g)^n\ , \tag{4-67}$$

where we use the notation $g = \rho - \frac{1}{2} + d/4$. Employing [13], formula 3.259.1, we reduce (4-67) to

$$4Kk\, \exp(-\lambda^2/2-2g) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(g\lambda^2)^n}{n!n!}\, \frac{(-2k)^m}{m!} \binom{n}{r}\left(\frac{k}{g}\right)^r B(3/2,\ 2m+2r+2)$$

$$= 4Kk\exp(-\lambda^2/2-2g) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(g\lambda^2)^{n+r}}{(n+r)!n!} \frac{(-2k)^m}{m!r!}\left(\frac{k}{g}\right)^r B(3/2,2m+2r+2)$$

$$= 4Kk\exp(-\lambda^2/2-2g) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{m} \frac{(g\lambda^2)^{n+r}}{(n+r)!n!} \frac{(-2k)^{m-r}}{(m-r)!r!}\left(\frac{k}{g}\right)^r B(3/2,2m+2)$$

$$= 4Kk\exp(-\lambda^2/2-2g) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(g\lambda^2)^n(-2k)^m}{n!(n+m)!} B(3/2,2m+2)\ L_m^n(\lambda^2/2), \quad (4\text{-}68)$$

a development analogous to (4-53). Thus for the mean value of the multiplicative array bearing estimate we have

$$E(\hat{\alpha}) = K - KQ(\lambda,\ 2\sqrt{g}) + 4Kk\exp(-\lambda^2/2-2g)$$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(g\lambda^2)^n(-2k)^m}{n!(n+m)!} B(3/2,2m+2)\ L_m^n(\lambda^2/2). \quad (4\text{-}69)$$

Going through a very similar process we find for the mean square,

$$E[(\hat{\alpha})^2] = K^2 - K^2Q(\lambda,\ 2\sqrt{g}) + 4K^2k\exp(-\lambda^2/2-2g)$$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(g\lambda^2)^n(-2k)^m}{n!(n+m)!} B(2,2m+2)\ L_m^n(\lambda^2/2). \quad (4\text{-}70)$$

The notation used in (4-69) and (4-70) is summarized by

$$K = 4.9/\beta\sqrt{4M^2-1} \qquad k = MH^2/B$$

$$\lambda^2 = [B^2+2C^2+4MH^2D^2(\tfrac{1}{2}\beta\sin\alpha;\ M)(B+C\cos 2\theta)]/B^2$$

$$d = 2 + \lambda^2 - 4MH^2D^2(\tfrac{1}{2}\beta\sin\alpha;\ M)\cos 2\theta/B-4C/B \quad (4\text{-}71)$$

$$g = d/4 - .5 + C/B = \lambda^2/4 - MH^2D^2(\tfrac{1}{2}\beta\sin\alpha;\ M)\cos 2\theta/B.$$

4.4.3 COMPUTED RESULTS

Specific cases of the probability density functions of the estimates (4-10) and (4-48) are given in Figures 4-6 and 4-7 for several values of the actual bearing, computed according to (4-50). At first sight what seems remarkable about these curves is that the bearing estimates at which the peak values are located do not correspond closely to the actual bearing. However, this in itself is not a reliable indication that the mean values diverge from what is desirable -- an unbiased estimate -- since there is a discrete probability that the value of the estimate is zero in each case. For the multiplicative array bearing estimate, there is also a discrete probability that the estimate equals its maximum value, arcsinK, not shown in the figure.

More revealing are the results for the mean values of the estimates, shown in Figures 4-8 and 4-9 versus actual bearing, with a straight line representing an unbiased estimator drawn in for reference. The trend, as the SNR increases, toward unbiasedness is clearly seen. This is comforting, since the theory of maximum likelihood estimation predicts this pehnomenon. These computations, based on (4-53) and (4-69), required on the order of 50 terms of each summation for H^2 = 1,2 and 100 terms for H^2 = 4.

The curves drawn with dashed lines in Figure 4-9 represent the series term in (4-69), so that the importance of the Q term is seen.

It appears from our limited computations that the standard and the multiplicative array estimators on the whole perform equally well with respect to bias. A kind of "end effect" is observed in either case

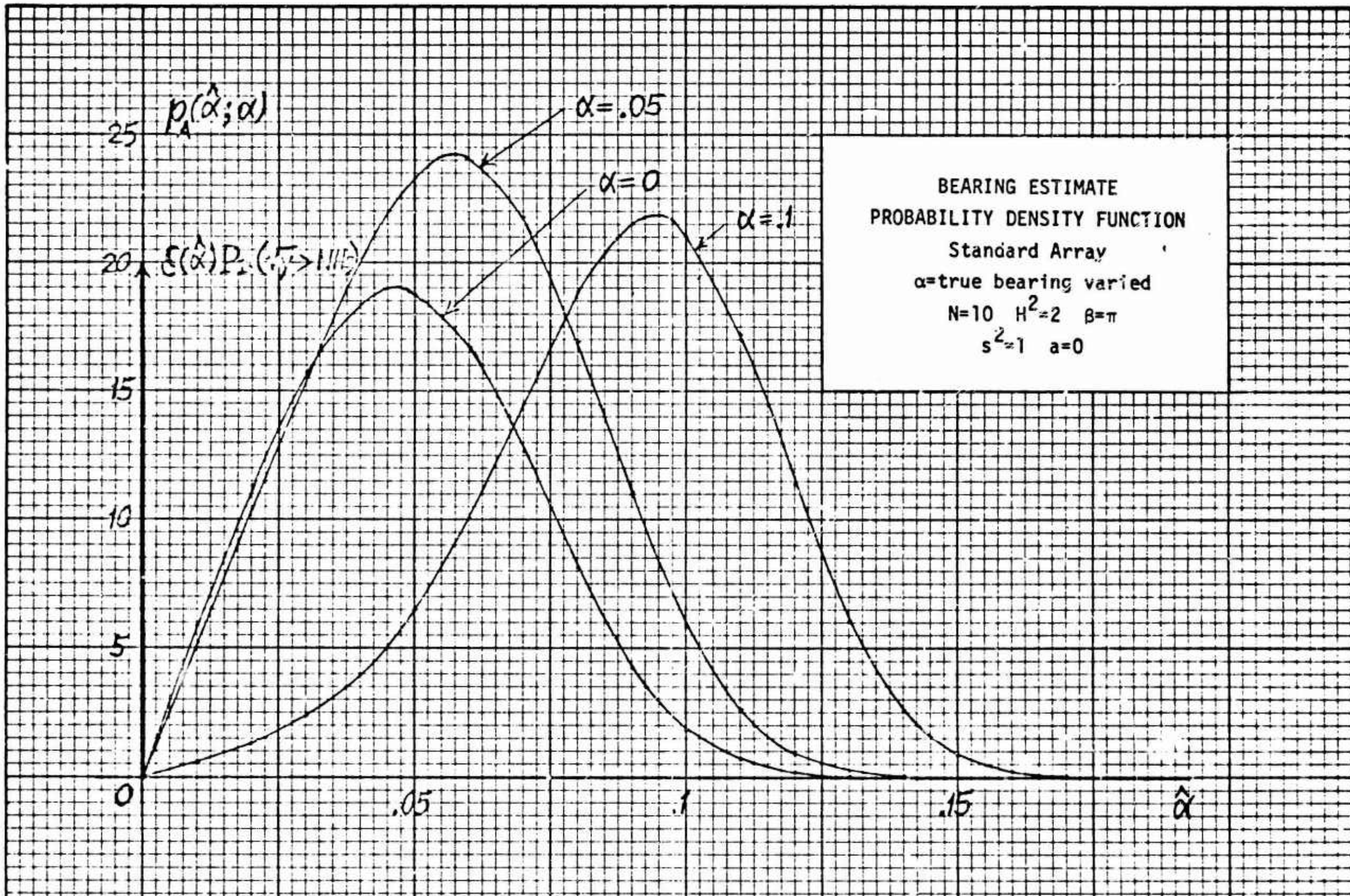

$p_A(\hat{\alpha};\alpha)$
$\alpha=.05$
$\alpha=0$
$\alpha=.1$
BEARING ESTIMATE
PROBABILITY DENSITY FUNCTION
Standard Array
α=true bearing varied
N=10 $H^2=2$ β=π
$s^2=1$ a=0
25
20
15
10
5
0
.05
.1
.15
$\hat{\alpha}$

FIGURE 4-6

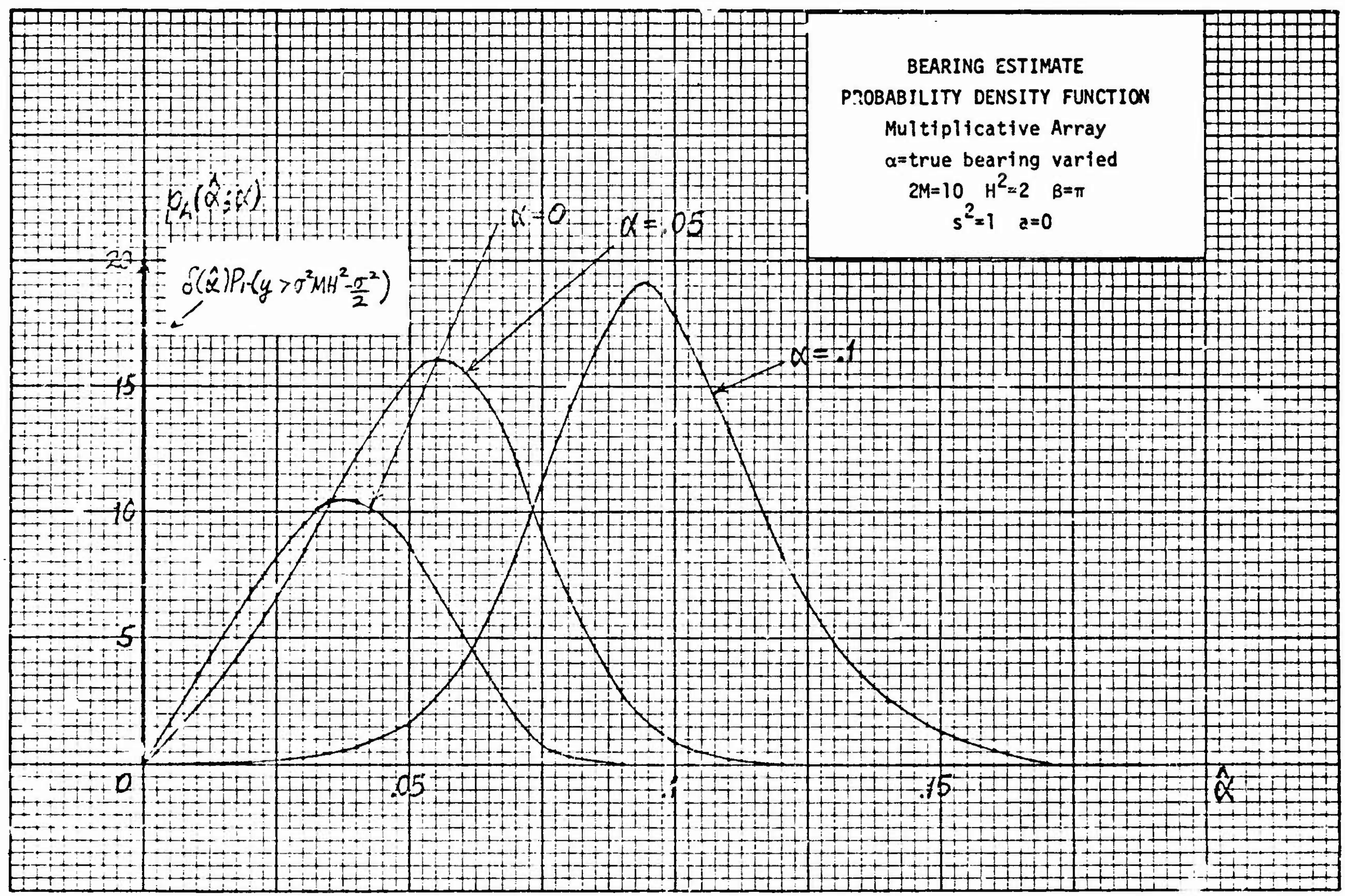
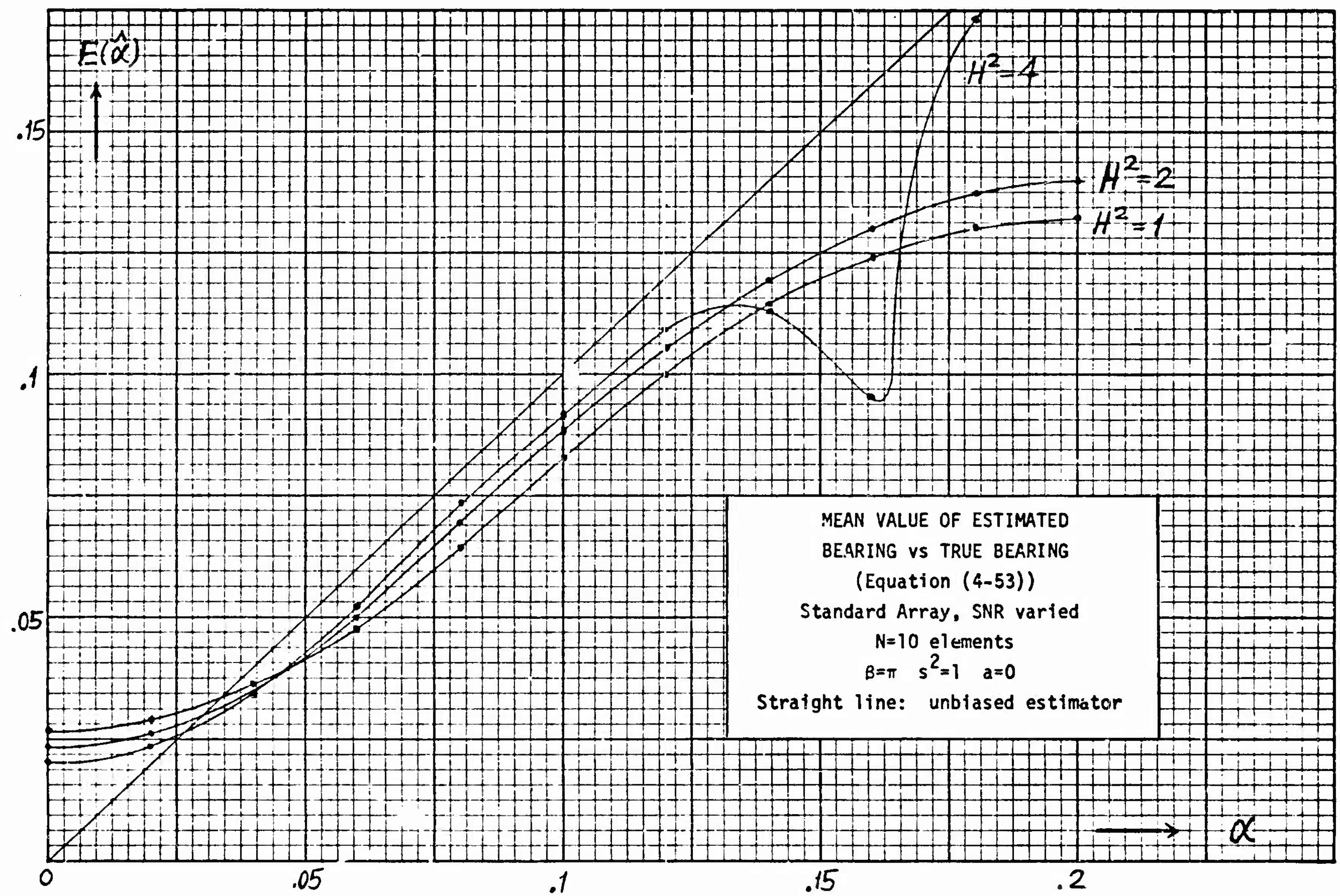
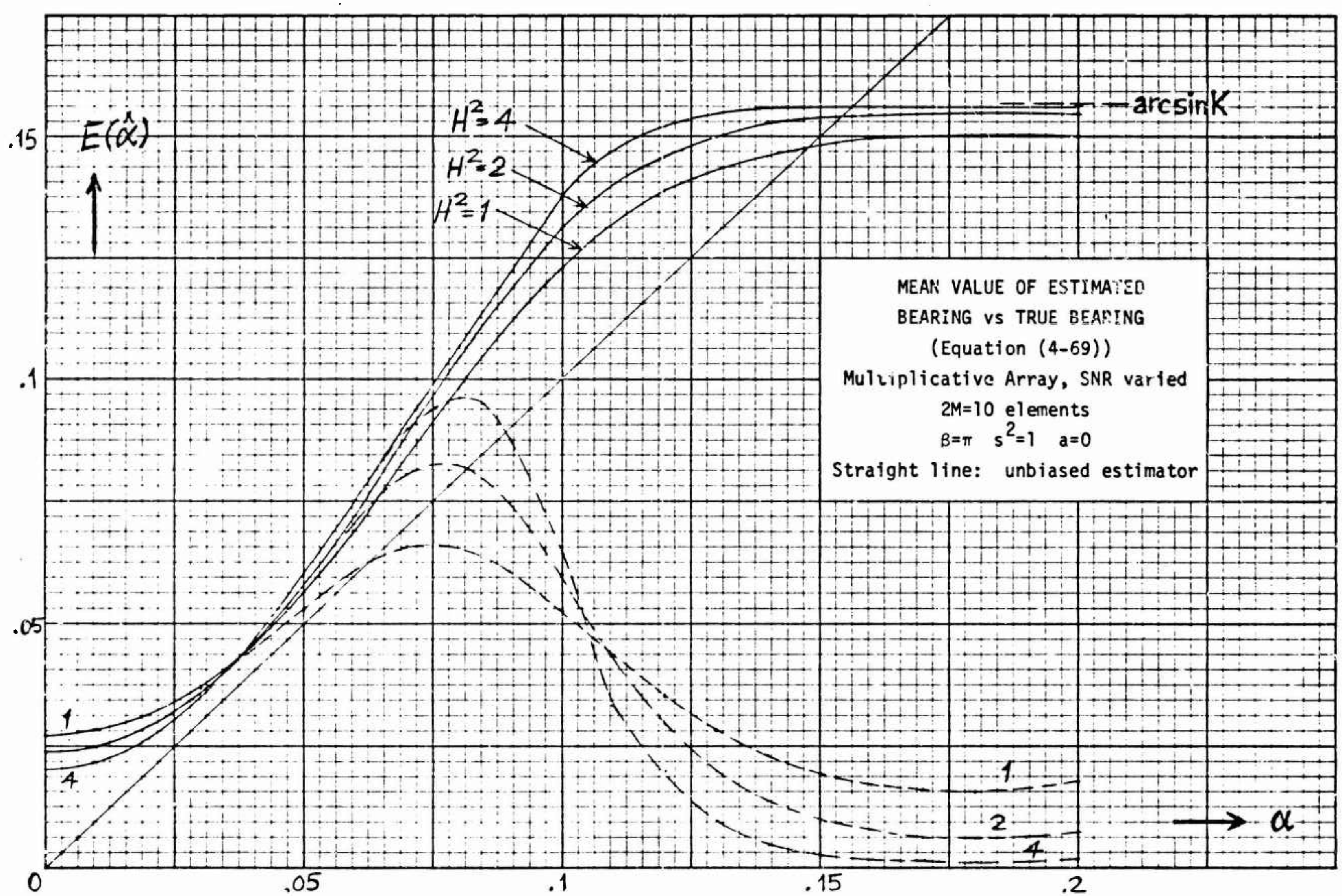

BEARING ESTIMATE
PROBABILITY DENSITY FUNCTION
Multiplicative Array
α=true bearing varied
2M=10 H²=2 β=π
s²=1 a=0
$p_L(\hat{\alpha};\alpha)$
$\delta(\hat{\alpha})P_1(y > \sigma^2 MH^2 - \frac{\sigma^2}{2})$
α=0
α=.05
α=.1
20
15
10
5
0
.05
.1
.15
$\hat{\alpha}$

FIGURE 4-7



$E(\hat{\alpha})$
.15
.1
.05
0
.05
.1
.15
.2
α
$H^2=4$
$H^2=2$
$H^2=1$
MEAN VALUE OF ESTIMATED
BEARING vs TRUE BEARING
(Equation (4-53))
Standard Array, SNR varied
N=10 elements
$\beta=\pi$ $s^2=1$ a=0
Straight line: unbiased estimator

FIGURE 4-8



MEAN VALUE OF ESTIMATED
BEARING vs TRUE BEARING
(Equation (4-69))
Multiplicative Array, SNR varied
2M=10 elements
β=π s²=1 a=0
Straight line: unbiased estimator
E(α̂)
α
arcsinK
H²=4
H²=2
H²=1
1
2
4
0
.05
.1
.15
.2

FIGURE 4-9

when $\hat{\alpha}$ approaches its maximum value. The difference is that the standard array estimator behaves smoothly in this regard, while the multiplicative estimator displays a more drastic reaction to this limitation, imposed by the formulation (4-63) of the estimator. It should be remembered that, if our attention is mainly upon bearings within the major lobe (α < arcsin .2 for the standard, α < arcsin .1 for the multiplicative), then the important region is that near $\alpha = 0$.

Computations of mean square, variance, and mean square error are shown in Figures 4-10 and 4-11. From the curves it is evident that with respect to these measures, the standard array estimator performs better than the multiplicative for increasing bearing.

Further consideration of estimator variance and mean square error is given in the next chapter, where we compare them with theoretical bounds.

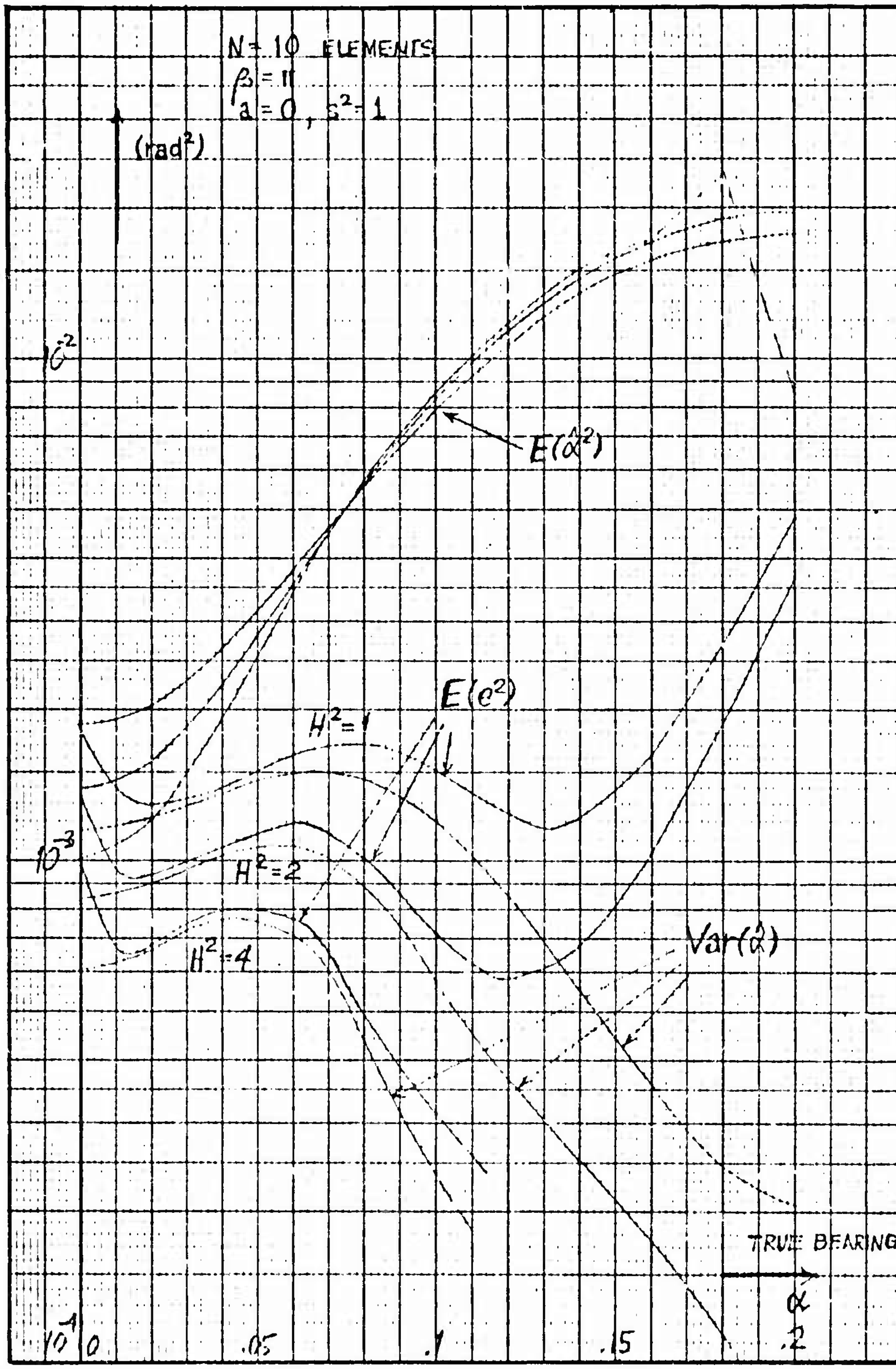


FIGURE 4-10 ESTIMATOR MEAN SQUARES AND VARIANCES, SQUARE-LAW ARRAY

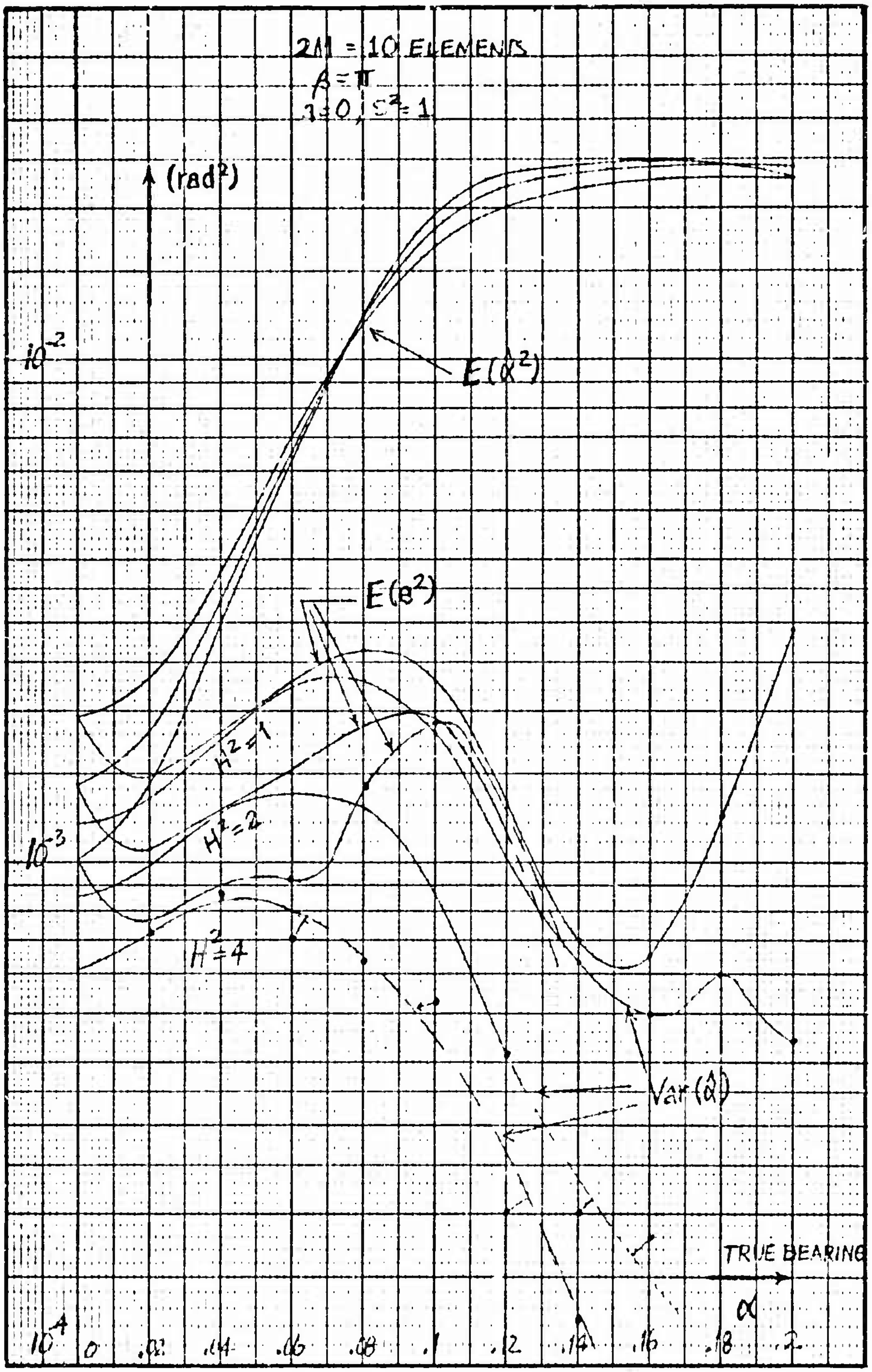


FIGURE 4-11 ESTIMATOR MEAN SQUARES AND VARIANCES, MULTIPLICATIVE ARRAY

CHAPTER FIVE

BEARING ESTIMATION PERFORMANCE

As mentioned in the introduction to Chapter 4, in addition to characterizing the detection performance of the two array models we are studying, we also desire to evaluate the bearing estimation capabilities of these configurations. That is, we should like to know with what potential precision the filter outputs of these array processing models can be operated upon to yield a value for the bearing of the signal source.

In the previous chapter we derived several forms for maximum likelihood bearing estimators, based on the probability density functions at the filter outputs which were found in Chapter 2. In this chapter we shall examine the performance of these estimators in terms of the relationship between the theoretical minimum estimator variance and the actual variance of the given estimator, as a function of SNR.

Although there are other, more exact bounds [34,38], we shall be making use of the Cramer-Rao bound on the variance of an estimator:

$$\mathrm{Var}(\hat{\alpha}) \geq \left[\frac{\partial}{\partial\alpha}E(\hat{\alpha})\right]^2 \Big/ E\left[\left\{\frac{\partial \ln p(\underline{w};\ \alpha)}{\partial\alpha}\right\}^2\right] \qquad (5\text{-}1)$$

where $p(\underline{w};\ \alpha)$ represents the distribution on which the estimate of the parameter is based [16], and where certain regularity conditions on the probability density [39] hold.

Bounds will first be calculated based on the Gaussian distributions seen by the array elements, and also those gaussian distributions holding at the array sums. Then bounds based on the filter output distributions will be calculated. These latter bounds we shall refer to in a general way as noncentral chi-square bounds.

5.1 CRAMER-RAO BOUNDS FOR GAUSSIAN DISTRIBUTIONS

If, analogous to Section 2.1.1, we annotate the N samples $\{w_k\}$ of a signal-plus-noise process by the vector $\underline{w}$ and the corresponding signal components $\{x_k\}$ by $\underline{x}$, the joint distribution for the Gaussian case is

$$p(\underline{w};\, \alpha) = \frac{1}{(2\pi)^{N/2}\sqrt{\det K_N}} \exp\left[-\tfrac{1}{2}(\underline{w}-\underline{x})' K_N^{-1} (\underline{w}-\underline{x})\right], \tag{5-2}$$

where α is the parameter of interest and $K_N = E[(\underline{w}-\underline{x})(\underline{w}-\underline{x})']$ is the covariance matrix of the samples, which is symmetric. For this expression of the distribution, we have

$$\frac{\partial \ln p(\underline{w};\, \alpha)}{\partial \alpha} = \left(\frac{\partial \underline{x}}{\partial \alpha}\right)' K_N^{-1} (\underline{w}-\underline{x}) \tag{5-3}$$

and

$$\left[\frac{\partial \ln p(\underline{w};\, \alpha)}{\partial \alpha}\right]^2 = \left(\frac{\partial \underline{x}}{\partial \alpha}\right)' K_N^{-1} (\underline{w}-\underline{x})(\underline{w}-\underline{x})' K_N^{-1} \left(\frac{\partial \underline{x}}{\partial \alpha}\right) \tag{5-4}$$

so that the expectation of (5-4) is given by

$$\left(\frac{\partial \underline{x}}{\partial \alpha}\right)' K_N^{-1} E[(\underline{w}-\underline{x})(\underline{w}-\underline{x})'] K_N^{-1} \left(\frac{\partial \underline{x}}{\partial \alpha}\right) = \left(\frac{\partial \underline{x}}{\partial \alpha}\right)' K_N^{-1} \left(\frac{\partial \underline{x}}{\partial \alpha}\right). \tag{5-5}$$

5.1.1 APPLICATION TO THE ARRAY INPUTS

If the N hydrophone inputs are considered in their narrowband, quadrature representations, then $\underline{w}$ in (5-1) is a vector with 2N components:

$$\underline{w} = \begin{bmatrix} \underline{w}_c \\ \\ \underline{w}_s \end{bmatrix} = \begin{Bmatrix} n_{kc} + x_{kc} \\ \\ n_{ks} + x_{ks} \end{Bmatrix} \tag{5-6}$$

where, if the phase reference is the center of the array,

$$\begin{aligned} x_{kc} &= A\cos\left[\tfrac{1}{2}(2k-N-1)\beta\sin\alpha\right] \\ x_{ks} &= A\sin\left[\tfrac{1}{2}(2k-N-1)\beta\sin\alpha\right] \end{aligned} \qquad k = 1,2,\ldots N. \tag{5-7}$$

The covariance matrix is 2N x 2N, with the form

$$K_{2N} = \begin{bmatrix} E\,[(\underline{w}_c-\underline{x}_c)(\underline{w}_c-\underline{x}_c)'] & E\,[(\underline{w}_c-\underline{x}_c)(\underline{w}_s-\underline{x}_s)'] \\ \\ E\,[(\underline{w}_s-\underline{x}_s)(\underline{w}_c-\underline{x}_c)'] & E\,[(\underline{w}_s-\underline{x}_s)(\underline{w}_s-\underline{x}_s)'] \end{bmatrix} \tag{5-8}$$

For white noise, K_{2N} is a diagonal matrix; further, if (as we have been assuming) the noise variances at the elements are equal, then $K_{2N} = s^2 I_{2N}$, so that $K_{2N}^{-1} = I_{2N}/s^2$, where I_{2N} is the 2N x 2N identity matrix. For this case, equation (5-5) becomes

$$\left(\frac{\partial x}{\partial \alpha}\right)' K_{2N}^{-1} \left(\frac{\partial x}{\partial \alpha}\right) = \frac{1}{s^2} \sum_{k=1}^{N} \left[\left(\frac{\partial x_{kc}}{\partial \alpha}\right)^2 + \left(\frac{\partial x_{ks}}{\partial \alpha}\right)^2\right] \tag{5-9}$$

$$= \frac{A^2\beta^2\cos\alpha^2}{s^2} \sum_{k=1}^{N} \left(k - \frac{N+1}{2}\right)^2 \tag{5-10}$$

$$= \frac{1}{6} N(N^2-1)H^2\beta^2\cos^2\alpha. \tag{5-11}$$

The form (5-10) is equivalent to that shown by Seidman [34] in the same case. The error of a bearing estimate $\hat{\alpha}(w;\ \alpha)$ based on the array input noise distribution for white noise is bounded by

$$\mathrm{Var}\ (\hat{\alpha}) \geqslant 6[\tfrac{\partial}{\partial\alpha}E(\hat{\alpha})]^2/N(N^2-1)H^2\beta^2\cos^2\alpha. \tag{5-12}$$

5.1.2 APPLICATION TO THE ARRAY SUMS

For the standard array model, which has the single array sum (1-4), the quadrature components are

$$\begin{aligned} w_c &= n_c + NAD(\tfrac{1}{2}\beta\sin\alpha;\ N)\cos[\tfrac{1}{2}(N-1)\beta\sin\alpha] \\ w_s &= n_s + NAD(\tfrac{1}{2}\beta\sin\alpha;\ N)\sin[\tfrac{1}{2}(N-1)\beta\sin\alpha], \end{aligned} \tag{5-13}$$

where n_c and n_s are modeled as independent Gaussian variates with variances $\sigma^2 = NB's^2$ (Section 2.2.2). Thus (5-5) in this case is

$$\begin{aligned} &\frac{N^2A^2\beta^2\cos^2\alpha}{4\sigma^2}\Big[\Big\{D'\cos[\tfrac{1}{2}(N-1)\beta\sin\alpha] - (N-1)D\sin[\tfrac{1}{2}(N-1)\beta\sin\alpha]\Big\}^2 \\ &\qquad + \Big\{D'\sin[\tfrac{1}{2}(N-1)\beta\sin\alpha] + (N-1)D\cos[\tfrac{1}{2}(N-1)\beta\sin\alpha]\Big\}^2\Big] \\ &= \frac{NH^2\beta^2\cos^2\alpha}{2B'}\ [(D')^2 + (N-1)^2D^2], \end{aligned} \tag{5-14}$$

so that an estimate $\hat{\alpha}(w_c, w_s:\ \alpha)$ based on the standard array sum has the bound

$$\mathrm{Var}\ (\hat{\alpha}) \geqslant 2B'\ [\tfrac{\partial}{\partial\alpha}E(\hat{\alpha})]^2/NH^2\beta^2\cos^2\alpha[(D')^2 + (N-1)^2D^2]. \tag{5-15}$$

For the two array sums of the multiplicative array model, the quadrature noise components have the covariance matrix given by (1-7).

Also, from (1-5) we have

$$x_{1c} = S\cos\theta_1, \; x_{1s} = S\sin\theta_1$$
$$x_{2c} = S\cos\theta_2, \; x_{2s} = S\sin\theta_2 \qquad (5\text{-}16)$$

where
$$S = MAD(\tfrac{1}{2}\beta\sin\alpha; M)$$
$$\theta_1 = \tfrac{1}{2}(M-1)\beta\sin\alpha, \; \theta_2 = \tfrac{1}{2}(3M-1)\beta\sin\alpha \qquad (5\text{-}17)$$

For what we have called our main case, the quadratic form (5-5) is

$$\frac{1}{\sigma^2(1-\rho^2)}\left\{\left(\frac{\partial x_{1c}}{\partial\alpha}\right)^2 + \left(\frac{\partial x_{1s}}{\partial\alpha}\right)^2 + \left(\frac{\partial x_{2c}}{\partial\alpha}\right)^2 + \left(\frac{\partial x_{2s}}{\partial\alpha}\right)^2 -2\rho\left[\left(\frac{\partial x_{1c}}{\partial\alpha}\right)\left(\frac{\partial x_{2c}}{\partial\alpha}\right) + \left(\frac{\partial x_{1s}}{\partial\alpha}\right)\left(\frac{\partial x_{2s}}{\partial\alpha}\right)\right]\right\} \qquad (5\text{-}18)$$

$$= \frac{M^2A^2\beta^2\cos^2\alpha}{4\sigma^2(1-\rho^2)}\Big\{[D'\cos\theta_1-(M-1)D\sin\theta_1]^2 + [D'\sin\theta_1+(M-1)D\cos\theta_1]^2$$
$$+ [D'\cos\theta_2-(3M-1)D\sin\theta_2]^2 + [D'\sin\theta_2+(3M-1)D\cos\theta_2]^2$$
$$- 2\rho[D'\cos\theta_1-(M-1)D\sin\theta_1][D'\cos\theta_2-(3M-1)D\sin\theta_2]$$
$$- 2\rho[D'\sin\theta_1+(M-1)D\cos\theta_1][D'\sin\theta_2+(3M-1)D\cos\theta_2]\Big\} \qquad (5\text{-}19)$$

$$= \frac{MH^2\beta^2\cos^2\alpha}{2B(1-\rho^2)}\Big\{2(D')^2+[(M-1)^2+(3M-1)^2]D^2$$
$$-2\rho\cos 2\theta[(D')^2+(M-1)(3M-1)D^2]-4\rho MDD'\sin 2\theta\Big\}, \qquad (5\text{-}20)$$

where $2\theta = \theta_1 - \theta_2$. If the phase reference is made the center of the array, that is if

$$\theta_1 = -\theta_2 = \tfrac{1}{2}(2M-1)\beta\sin\alpha, \tag{5-21}$$

then (5-5) becomes

$$\frac{MH^2\beta^2\cos^2\alpha}{2B(1-\rho^2)}\; 2[(D')^2+(2M-1)^2D^2](1-\rho\cos 2\theta)$$

$$= \frac{MH^2\beta^2\cos^2\alpha}{B^2-C^2}\;[(D')^2+(2M-1)^2D^2](B-C\cos M\beta\sin\alpha). \tag{5-22}$$

Therefore, an estimate $\hat{\alpha}(w;\alpha)$ based on the two array sums has an error bound,

$$\mathrm{Var}(\hat{\alpha}) \geqslant (B^2-C^2)[\tfrac{\partial}{\partial\alpha}E(\hat{\alpha})]^2/MH^2\beta^2\cos^2\alpha[(D')^2+(2M-1)^2D^2](B-C\cos M\beta\sin\alpha). \tag{5-23}$$

5.1.3 COMPARISON OF BOUNDS AT ARRAY SUMS

For unbiased estimates, that is for $E(\hat{\alpha}) = \alpha$, we may compare the minimum estimator variances of the two array configurations by comparing (5-15) and (5-23). Let the minimum variances of the standard and multiplicative arrays be denoted $\sigma^2_{min,s}$ and $\sigma^2_{min,m}$ respectively. Then, for the same number of elements (N=2M),

$$\frac{\sigma^2_{min,s}}{\sigma^2_{min,m}} = \frac{(D'_{N/2})^2+(N-1)^2D^2_{N/2}}{(D'_N)^2 + (N-1)^2\, D^2_N} \cong \frac{D^2_{N/2}}{D^2_N}\,, \tag{5-24}$$

where $D_K = D(\tfrac{1}{2}\,\beta\sin\alpha;\, K)$. Since $D_{N/2} \geqslant D_N$ for the angles of interest, we have

$$\sigma^2_{min,s} \geqslant \sigma^2_{min,m} \tag{5-25}$$

at the array sums, with equality holding at zero bearing. For example, for N = 10, $\beta = \pi$, and $\alpha = .05$, the ratio is .948/.808 = 1.24.

5.2 CRAMER-RAO BOUND FOR NONCENTRAL CHI-SQUARE DISTRIBUTIONS

Let x be a noncentral chi-square random variable with two degrees of freedom and noncentrality parameter λ^2. Then the probability density function of x is

$$p(x) = (1/2)\exp[-(x+\lambda^2)/2]\; I_0(\lambda\sqrt{x}), \tag{5-26}$$

where $\lambda = \lambda(\alpha)$, α a parameter to be estimated. For this density function,

$$\frac{\partial \ln p(x;\,\alpha)}{\partial\alpha} = -\lambda\lambda' + \frac{I_1(\lambda\sqrt{x})}{I_0(\lambda\sqrt{x})}\,\lambda'\sqrt{x} \tag{5-27}$$

so that $$E\left\{\left(\frac{\partial \ln p}{\partial\alpha}\right)^2\right\} = (\lambda')^2\left(\lambda^2 - 2\lambda E\left\{\sqrt{x}\,\frac{I_1(\lambda\sqrt{x})}{I_0(\lambda\sqrt{x})}\right\} + E\left\{x\left[\frac{I_1(\lambda\sqrt{x})}{I_0(\lambda\sqrt{x})}\right]^2\right\}\right)$$

$$= (\lambda')^2\left\{\lambda^2 - e^{-\lambda^2/2}\int_0^\infty dx\sqrt{x}\; e^{-x/2} I_1(\lambda\sqrt{x})\right.$$

$$\left. + (1/2)e^{-\lambda^2/2}\int_0^\infty x\,dx\; e^{-x/2}\,\frac{I_1(\lambda\sqrt{x})}{I_0(\lambda\sqrt{x})}\, I_1(\lambda\sqrt{x})\right\}. \tag{5-28}$$

From Gradshteyn and Ryzhik ([13], formulas 6.643.2, 9.220.2) and Middleton ([16], formula A.1.19b) we have for the first integral,

$$\int_0^\infty dx\sqrt{x}\; e^{-x/2}\; I_1(\lambda\sqrt{x}) = 2\lambda\, {}_1F_1(1,2;\,\lambda^2/2) = -\frac{4}{\lambda}\,(1-e^{\lambda^2/2}). \tag{5-29}$$

For the second integral we have approximately,

$$\int_0^\infty dx\; x\, e^{-x/2} \frac{I_1(\lambda\sqrt{x})}{I_0(\lambda\sqrt{x})} I_1(\lambda\sqrt{x}) \cong \int_0^\infty dx\; x\, e^{-x/2} I_1(\lambda\sqrt{x}) \tag{5-30}$$

$$= \frac{3}{2}\lambda\sqrt{2\pi}\; {}_1F_1(5/2,2;\, \lambda^2/2)$$

$$= \lambda\sqrt{2\pi}\left[\frac{1}{2}\, {}_1F_1(1/2,2;\, \lambda^2/2) + (1+\lambda^2/2)\, {}_1F_1(3/2,2;\, \lambda^2/2)\right]$$

$$= \frac{\lambda}{2}\sqrt{2\pi}\; e^{\lambda^2/4}[(3+\lambda^2)I_0(\lambda^2/4) + (1+\lambda^2)I_1(\lambda^2/4)], \tag{5-31}$$

again making use of [13] and [16].

Therefore, (5-28) becomes

$$E\left\{\left(\frac{\partial \ln p}{\partial \alpha}\right)^2\right\} = (\lambda')^2\Big\{\lambda^2 + 4e^{-\lambda^2/2}\,(1-e^{\lambda^2/2})$$

$$+ \frac{\lambda}{4}\sqrt{2\pi}\; e^{-\lambda^2/4}[(3+\lambda^2)I_0(\lambda^2/4) + (1+\lambda^2)I_1(\lambda^2/4)]\Big\}. \tag{5-32}$$

For large arguments of the Bessel functions (or large λ) we may write,

$$E\left\{\left(\frac{\partial \ln p}{\partial \alpha}\right)^2\right\} = (\lambda')^2(2\lambda^2-2+4e^{-\lambda^2/2}) \cong 2(\lambda')^2(\lambda^2-1). \tag{5-33}$$

5.2.1 CRAMER-RAO BOUND FOR THE STANDARD ARRAY FILTER OUTPUT

For the standard array filter output, which is distributed as a noncentral chi-square variable, we have

$$\lambda^2 = 2h^2 = 2NH^2D^2(\tfrac{1}{2}\beta\sin\alpha;\ N)/B'. \tag{5-34}$$

Thus the denominator of the Cramer-Rao bound expression (5-1) is

$$E\left\{\left(\frac{\partial \ln p_{sq}(y)}{\partial\alpha}\right)^2\right\} = 4(h')^2(2h^2-1)$$

$$= 2N^3H^2\beta^2\cos^2\alpha(D')^2(2NH^2D^2-B')/(B')^2. \tag{5-35}$$

From (4-53) we may write, using [13], formula 8.971,

$$\frac{\partial}{\partial\alpha}E(\hat{\alpha}) = -2Kk^2DD'N\beta\cos\alpha e^{-kD^2}\sum_{m=0}^{\infty}\frac{(-k)^m}{m!}B(3/2,2m+2)[L_m+L_{m-1}^1]$$

$$= -2Kk^2DD'\beta\cos\alpha e^{-kD_N^2}\sum_{m=0}^{\infty}\frac{(-k)^m}{m!}B(3/2,2m+2)\ L_m^1(kD^2), \tag{5-36}$$

since $h^2 = kD^2$, so that

$$\left[\frac{\partial}{\partial\alpha}E(\hat{\alpha})\right]^2 = 4K^2k^4D^2(D')^2N^2\beta^2\cos^2\alpha\ e^{-2kD^2}$$

$$\times\left\{\sum_{m=0}^{\infty}\frac{(-k)^m}{m!}B(3/2,2m+2)\ L_m^1(kD^2)\right\}^2. \tag{5-37}$$

The Cramer-Rao bound for the standard array filter output is then the ratio of (5-37) to (5-35), using Σ to denote the series:

$$\mathrm{Var}(\hat{\alpha}) \geq 2K^2N^3H^6D^2(\ \Sigma\ )^2e^{-2ND^2H^2/B'}\ /(B')^2(2NH^2D^2-B')$$

$$\cong 24N^2H^4(\ \Sigma\ )^2e^{-2ND^2H^2/B'}\ /(B')^2\beta^2(N^2-1). \tag{5-38}$$

For white noise (B' = 1) this expression becomes

$$\mathrm{Var}(\hat{\alpha}) \geqslant 24N^2H^4(\Sigma_w)^2 e^{-2ND^2H^2}/\beta^2(N^2-1), \tag{5-39}$$

and when the bearing is zero also,

$$\mathrm{Var}(\hat{\alpha}) \geqslant 24N^2H^4(\Sigma_{w,0})^2 e^{-2NH^2}/\beta^2(N^2-1). \tag{5-40}$$

5.2.2 CRAMER-RAO BOUND FOR THE MULTIPLICATIVE ARRAY FILTER OUTPUT

For the multiplicative array filter output, which is distributed approximately as a noncentral chi-square variable, we have from (4-71) and Section 4.3.2,

$$\lambda^2 = 1 + 2\rho^2 + 4h^2(1+\rho\cos 2\theta)$$

$$= [B^2 + 2C^2 + 4MH^2D^2(\tfrac{1}{2}\beta\sin\alpha;\, M)(B+C\cos M\beta\sin\alpha)]/B^2. \tag{5-41}$$

The "x" in (5-26) corresponds here to $4y/\sigma^2+d$, where y is the filter output. The expression (5-33) may be used here if we can say that $\partial x/\partial\alpha = 0$, as assumed in (5-27) and in the following analysis. But

$$\frac{\partial x}{\partial\alpha} = \frac{\partial d}{\partial\alpha} = \frac{\partial}{\partial\alpha}[3 - 4\rho + 2\rho^2 + 4h^2 - 4h^2(1-\rho)\cos 2\theta]$$

$$= 4M^2H^2D\beta\cos\alpha\,\{D'[B-(B-C)\cos 2\theta] + D(B-C)\sin 2\theta\}/B^2. \tag{5-42}$$

For white noise (B = 1, C = 0) this relation is

$$\frac{\partial d}{\partial\alpha} = 8M^2H^2D\beta\cos\alpha\sin\theta(D'\sin\theta+D\cos\theta) \longrightarrow 0 \text{ as } \alpha \longrightarrow 0. \tag{5-43}$$

Thus, though we shall use (5-33), we do so with the understanding that the solution involves this additional approximation when the bearing is nonzero.

Proceeding to apply (5-33), then, we have

$$E\left\{\left(\frac{\partial \ln p(y)}{\partial \alpha}\right)^2\right\} = 2(\lambda')^2(\lambda^2-1) = 2[(\lambda^2)'/2\lambda]^2(\lambda^2-1)$$

$$= \frac{[4M^2H^2D\beta\cos\alpha(D'B+D'C\cos 2\theta-DC\sin 2\theta)]^2}{B^2+2C^2+4MH^2D^2(B+C\cos 2\theta)}$$

$$\times\ [2C^2+4MH^2D^2(B+C\cos 2\theta)]/2B^2. \tag{5-44}$$

For white noise,

$$E\left\{\left(\frac{\partial \ln p(y)}{\partial \alpha}\right)^2\right\} = \frac{32M^5H^6D^4\beta^2\cos^2\alpha(D')^2}{1+4MH^2D^2}\ . \tag{5-45}$$

Using (4-69), we may write

$$\frac{\partial}{\partial\alpha}E(\hat{\alpha}) = -K\lambda'\frac{\partial}{\partial\lambda}Q(\lambda,\ 2\sqrt{g}) - 4Kk\exp(-\lambda^2/2-2g)\ \lambda\lambda'$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-k)^m}{n!}B(3/2,2m+2)\ L_m^{n+1}(\lambda^2/2)\frac{(n+m+1-2g)}{(n+m+1)!}(g\lambda^2)^n. \tag{5-46}$$

where $g = d/4 + C/B - .5$. Representing the series by $\Sigma\Sigma$, we have

$$[\frac{\partial}{\partial\alpha}E(\hat{\alpha})]^2 = K^2(\lambda')^2[\frac{\partial}{\partial\lambda}Q(\lambda,\ 2\sqrt{g}) + 4k\lambda\exp(-\lambda^2/2-2g)\Sigma\Sigma]^2, \tag{5-47}$$

and the Cramer-Rao bound for the multiplicative array filter output may be expressed, using (5-44),

$$\mathrm{Var}(\hat{\alpha}) \geqslant \frac{K^2}{2(\lambda^2-1)}\left[\frac{\partial}{\partial\lambda}Q(\lambda,\ 2\sqrt{g}) + 4k\lambda\exp(-\lambda^2/2-2g)\Sigma\Sigma\right]^2, \tag{5-48}$$

where the notation is given by (4-71). For small angles, the Q term vanishes, yielding

$$\mathrm{Var}(\hat{\alpha}) \geq 8K^2k^2\lambda^2\exp(-\lambda^2-4g)(\Sigma\Sigma)^2/(\lambda^2-1). \tag{5-49}$$

For white noise we obtain for small angles,

$$\mathrm{Var}(\hat{\alpha}) \geq \frac{192M^2H^4}{\beta^2\sqrt{4M^2-1}}\,\frac{2+4MH^2D^2}{1+4MH^2D^2}\exp[-4-4MH^2D^2(2-\cos 2\theta)](\Sigma\Sigma)^2. \tag{5-50}$$

5.3 COMPUTED RESULTS

Computations of the various bounds on estimator error derived in this chapter are shown in the accompanying figures.

In Figure 5-1, we show diagrammatically how these bounds correspond to the density functions at different points in the array models. The bound at the array inputs (5-12) is based on the joint probability density function of the outputs of the array elements, and take into account the basic geometry of the array. The bounds (5-15) and (5-23) are based on the density functions of the array sums, and reflect knowledge of the directivity gained by the summing. And, the bounds (5-38) and (5-48) are based on the filter otuput probability density functions, and incorporate the effects of the nonlinear processing involved in detection.

This procedure of computing bounds at several points in a system is somewhat unusual, but was undertaken for two reasons: (1) after the summations, to provide a basis of comparison where the two array systems begin to differ; (2) after the filters, partly out of curiosity and partly to obtain a measure of the effects of the nonlinear processing. The effort was worthwhile if only for the interesting discussions the

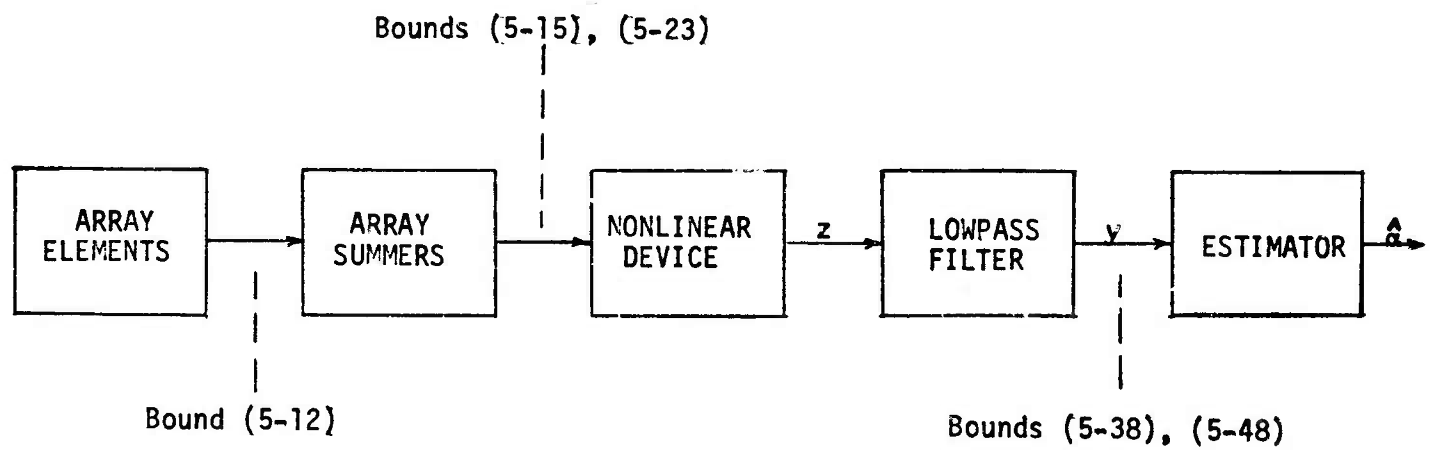


FIGURE 5-1 POINTS IN THE ARRAY SYSTEMS AT WHICH BOUNDS WERE CALCULATED

results have stimulated thus far, points which we shall mention in the following presentation of the numerical results.

Figure 5-2 displays the bounds (5-15) and (5-23) in comparison with (5-12) for estimators assumed to be unbiased, that is for which $E(\hat{\alpha}) = \alpha$, for $H^2 = 1$ and $H^2 = 2$, versus actual bearing. The arrays are assumed to have ten elements and half-wavelength spacing. There are two significant features of this figure. The first is that for small bearings the bounds computed from the density functions at the array sums are lower than that computed at the array inputs. In fact, if we compare (5-15) and (5-12) for zero bearing we find that

$$\frac{\text{Bound at sums}}{\text{Bound at inputs}} = \frac{1}{3}\,\frac{N+1}{N-1}\ , \ \alpha = 0, \tag{5-51}$$

where N is the number of elements. Thus for N = 2 we see that the bounds are indeed equivalent, so that for other values of N the comparison should be valid. On the other hand, given the manner in which the bearing is "encoded" or embedded in the signal terms and recalling the results obtained for estimator means in Chapter 4, we cannot push this comparison very far since it appears likely that estimates of bearing based on the summations will in general be biased.

The second feature to note in Figure 5-2 is that the bound (5-23) for the multiplicative array bearing estimate lies below that for the square-law or standard array (5-15). This has already been noted in Section 5.1.3, and implies that the accuracy of the multiplicative configuration's two summation (split-beam) approach is potentially better for bearing estimation. Also, the increasing error for higher angles typifies the cost of beamforming.

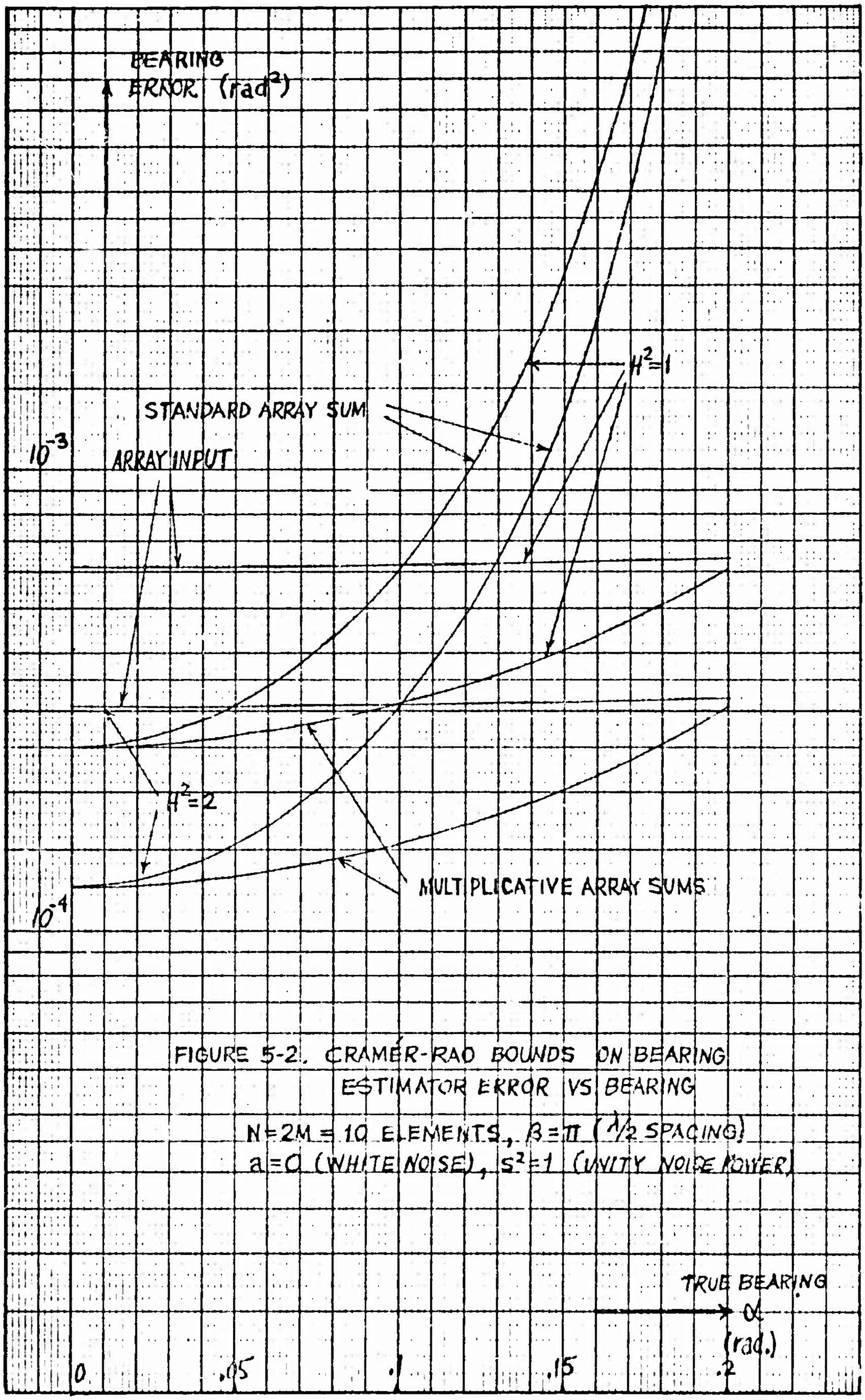


FIGURE 5-2. CRAMÉR-RAO BOUNDS ON BEARING ESTIMATOR ERROR VS BEARING

$N=2M=10$ ELEMENTS, $\beta=\pi$ ($\lambda/2$ SPACING)
$a=0$ (WHITE NOISE), $s^2=1$ (UNITY NOISE POWER)

Further comparisons among these bounds are illustrated in Figure 5-3, versus SNR. Again, the position of the bound corresponding to the array sums of the multiplicative array model relative to that of the standard model would seem to imply that the multiplicative array promises better performance for nonzero values of bearing, as noted in Section 5.1.3.

The somewhat difficult to compute noncentral chi-square bounds, (5-38) for the standard array estimator and (5-48) for the multiplicative, are shown versus true bearing in Figure 5-4. Since estimator bias is included in these bounds, these curves seem to be informing us that the combination of the nonlinear processing and the particular estimator forms we have selected promise much improved performance over that achievable at the array sums, for this range of bearings. However, it may be also, as Seidman implies in similar cases [34], that the Cramer-Rao bounding technique yields a loose (too small) bound for the error over the range of bearings and SNR in which we are interested.

Estimator error curves from Figures 4-10 and 4-11 are superimposed in Figure 5-5 versus true bearing. From their shape, it is obvious that these curves are "close relatives" to the post-filter bounds in Figure 5-4. However, it is also manifest that these estimators do not achieve the error performance implied by the post-filter bounds. Rather, the actual error curves are more closely bounded by the bounds of Figure 5-2. This occurrence, we believe, indicates the looseness of the post-filter bounds.

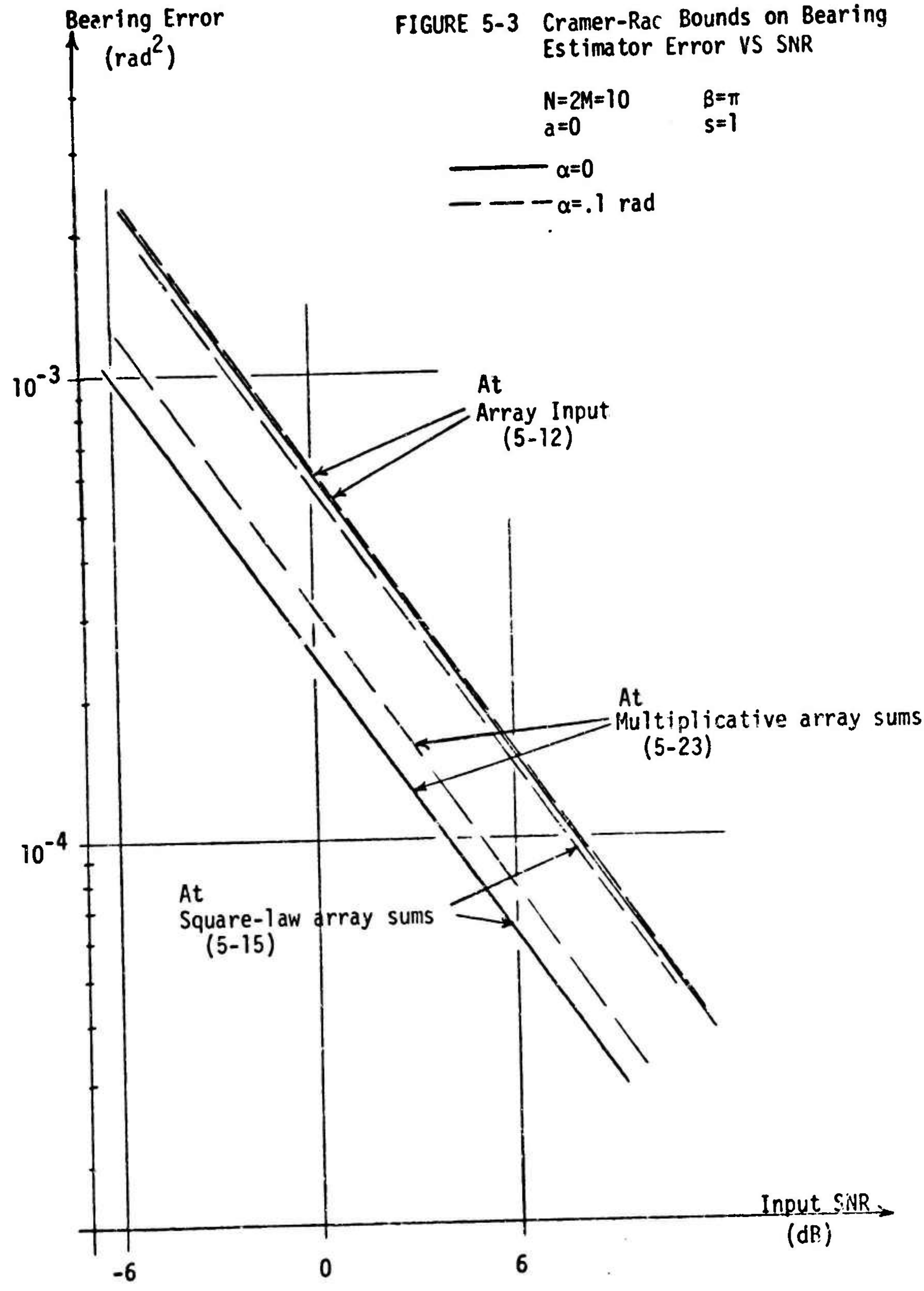


FIGURE 5-3 Cramer-Rac Bounds on Bearing Estimator Error VS SNR

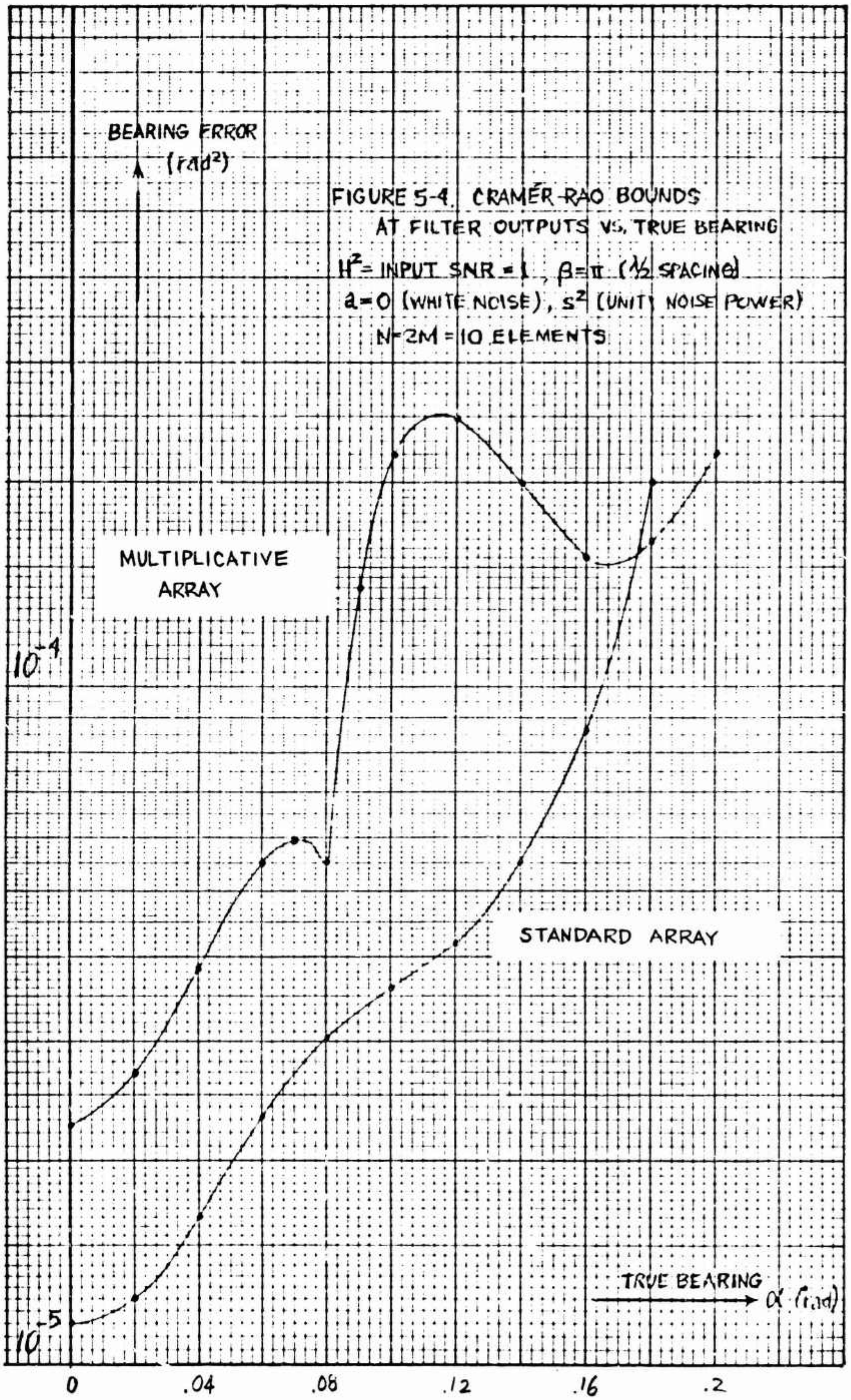


FIGURE 5-4. CRAMER-RAO BOUNDS AT FILTER OUTPUTS VS. TRUE BEARING

H^2 = INPUT SNR = 1, $\beta = \pi$ ($\lambda/2$ SPACING)

$a = 0$ (WHITE NOISE), s^2 (UNITY NOISE POWER)

$N = 2M = 10$ ELEMENTS

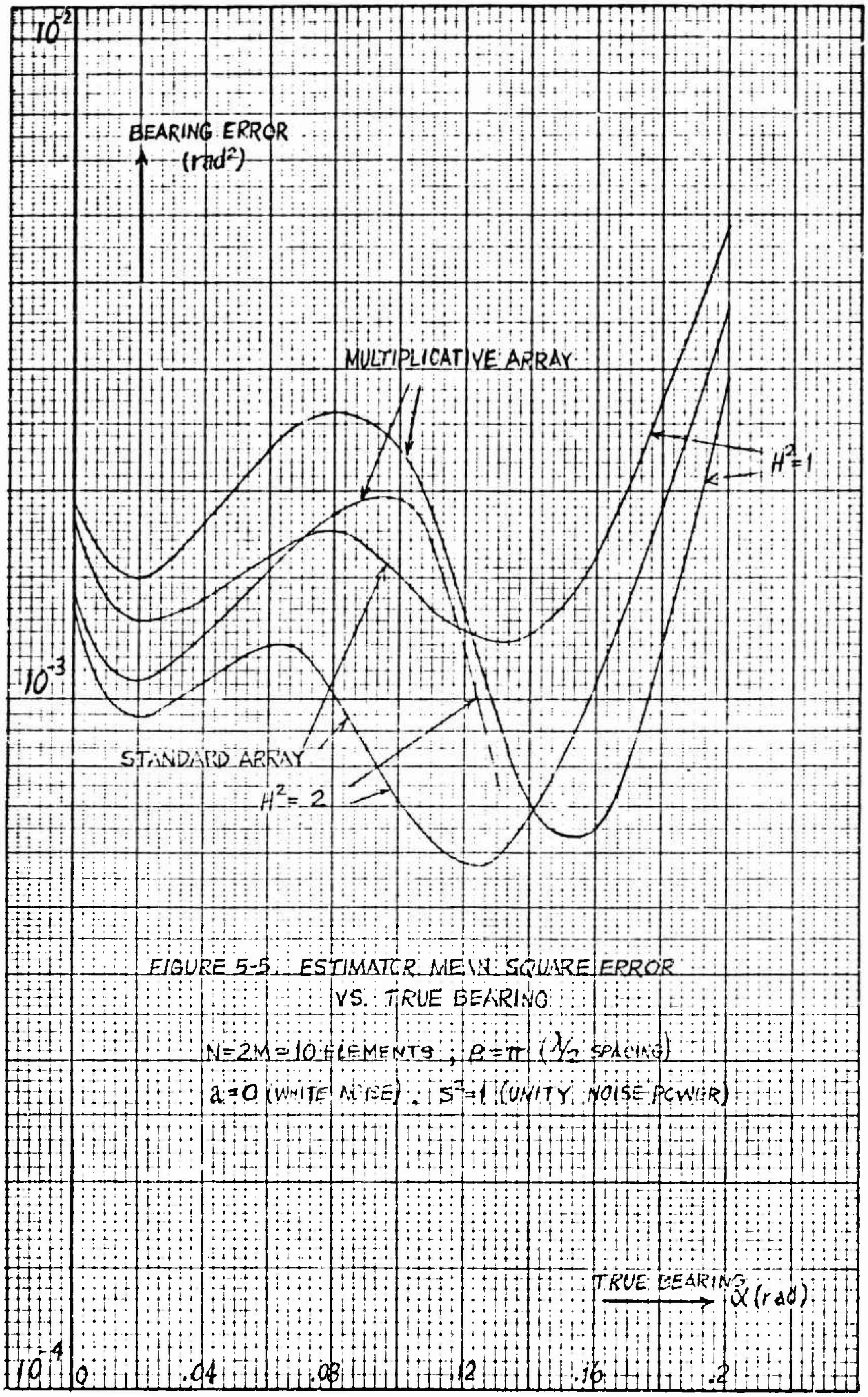


FIGURE 5-5. ESTIMATOR MEAN SQUARE ERROR VS. TRUE BEARING

$N=2M=10$ ELEMENTS ; $\beta=\pi$ ($\lambda/2$ SPACING)

$a=0$ (WHITE NOISE) ; $S^2=1$ (UNITY NOISE POWER)

Judging by the manner in which the actual estimator errors behave with increased SNR (decreasing not nearly as "fast" as the bounds), we may observe that the particular forms for the estimators, though obtained by means of the maximum likelihood method, are not asymptotically efficient. That is, from the data we have obtained, the error, though decreasing with higher SNR, does not converge toward the lower bound. Again, we remark that this may be due to the limitations of the bounding technique, but quite likely also the estimator forms, being approximations, are therefore suboptimum.

CHAPTER SIX

CONCLUSIONS

With this chapter we conclude this work and take stock of what has been done. First, the work is summarized, with particular emphasis upon what are considered to be the primary results. These comments are then qualified or put into perspective by a more critical review of the assumptions and methods employed. Finally, suggestions for improvements and extensions of the work are given.

6.1 SUMMARY

We began by describing a "debate" in the literature over the relative merits of what were termed the standard and the multiplicative processing of signals intercepted by linear arrays of receiving elements (hydrophones, for definiteness). Our conception of this debate was to that a considerable degree the issues are clouded by insufficient characterization of the random processes involved, in that some of the more important performance measures by which the two configurations ought to be compared are probabilistic in nature (e.g., probability of detection). The goal of this work, therefore, was set to provide at least a partial remedy to this situation by applying some of the analytical techniques of statistical communication theory to the problem, with a view toward system comparison.

Because a kind of "gain/bandwidth" tradeoff exists for analysis, rather simple narrowband models of standard, or square law, and multiplicative array processing systems were chosen so that full attention could be given to as rigorous an analysis of the performance of these systems as our abilities would permit. In this fashion we were enabled to include as generalizations the various array parameters, such as the number of elements, the inter-element spacing, and the noise covariances.

A rather general probability density function for the filter output of the narrowband array processor was derived by the direct method. This, we believe, is our basic result in the sense that the remainder of the work consists of application. The degree of generality is such that the effects of all the model parameters can be studied. Moreover, alternate computational forms were given for the main case, in which the array was considered symmetric and the noise assumed to have a spectrum even about the bandpass center frequency.

Theorizing a typical Neyman-Pearson characterization of signal detection at the filter outputs of the models, we were able to use the probability densities to calculate detection and false alarm probabilities and receiver operating characteristics. From the numerical results obtained, we observe that the standard detector or processor is only very slightly better than the multiplicative in terms of signal detection, so that we would assume that the better beamforming property of the multiplicative array gives it the edge in this category.

The more difficult area of application of the probability density functions was in exploring the bearing estimation capabilities of the arrays. We found ourselves unable to solve the general maximum likelihood equation for other than very small SNR. However, our exact knowledge of the distribution of the filter output made it possible to select as a good approximation the somewhat more tractable noncentral chi-square density function, with a linear function of the multiplicative array filter output as its argument. Forms of maximum likelihood estimators of bearing were shown for both high and low SNR cases for both array models.

Further treatment was given to these estimators for the high SNR cases, in the form of showing their means and mean squares, as well as in displaying example computations of their probability distributions. It was seen that these estimators do approach unbiasedness, although our modeling of the multiplicative array estimator apparently had a negative effect on its performance in this respect. Cramer-Rao bounds on the minimum estimator error were calculated for various points in the array systems, with the bound based on the density function at the array sums coming the closest to the actual mean square errors of the estimators. Although with respect to mean square error, the standard array estimator appeared to perform slightly better, it appeared also that for some ranges of bearing the multiplicative array estimator is better. Neither of the two specific forms for the estimators we selected was very efficient, however, an occurrence which we attribute to the amount of approximation involved in their derivation.

6.2 CRITIQUE

In keeping with our goal to shed light on the discussion of array performance, we now offer some critical comments, pro and con, on the information which we have generated to assist the reader in assessing it.

We believe that the derivation of the probability density functions we have shown represents an important contribution to the knowledge in this area, to the extent that the popular approach on which it is based (utilizing rather idealistic conceptions of lowpass filtering) is faithful to the behavior of narrowband systems. We should like to remind the reader that several of the parameters on which the distribution is conditioned, such as bearing and signal amplitude, might more realistically be treated as random variables. Also, the validity of assuming stationary Gaussian noise processes, although a popular procedure, is sometimes questionable for underwater applications.

To our knowledge, the detection analysis which we have offered is, in its exactness, also a "step input" to the narrow field for which it applies, although we would not be surprised to discover that similar data exists in various classified memoranda.

Most comparative detection analyses which have come to our attention, besides those cited in Chapter 1, rely upon detector output SNR as a figure of merit (e.g., [58-60]). While probability of detection is indeed proportional to SNR, we have demonstrated that comparison on the basis of SNR may exaggerate the advantage of the system having the greater SNR. However, this is not to discount this approach where the comparison involves detectors with the same type of output probability density function, when this property has been shown, as in optimization of a given configuration [61, 62].

The amount of data which we have shown is small, but we offer our computer programs in Appendix D as a cover for this deficiency, if it be such.

The numerical results of our investigation of maximum likelihood bearing estimation from the filter outputs are somewhat sparse, in proportion to the amount of labor spent in derivation. However, we are satisfied that the several theoretical expressions we have shown are useful accomplishments in themselves, going by the lack of assistance offered by the literature in this area. It is worth noting that though the feasibility of optimally estimating bearing using the filter outputs has been demonstrated, a more reasonable approach perhaps is to reconsider the entire configuration for these purposes. Stremler and Brown [54], for example, show a bearing estimating system using a phase detector. Also, split-beam and monopulse techniques are popular (e.g., [53, 58]).

6.3 SUGGESTIONS FOR FURTHER WORK

There are several directions which improvements and extensions of this work may take. We suggest a few examples:

-- Direct application: using the PDF's and detection analysis to evaluate similar systems, such as the split-beam tracker, for which the analysis either applies or can be modified to apply.

-- Further generalization: expanding the classes of array geometries; treating nondeterministic or unknown signals; considering joint estimation of bearing, SNR (see [55]), and perhaps range; treating nonstationary/nongaussian noise processes.

-- Improvements: using numerical methods to calculate more precise error bounds and to simulate more exact forms of the maximum likelihood estimator (see [56]); additional theoretical investigations to specify the system comparison problem over the broader range of bearings; a closer examination of the small SNR case.

APPENDIX A

DISCUSSION OF THE ARRAY INPUT NOISE PROCESS

Let the array input noise process be denoted $\nu(x,t)$ before steering and $n(x,t)$ after sterring, where x represents distance along the array, with x=0 at the center of the 2M elements. Let $\nu(x,t)$ be Gaussian, zero-mean, and stationary; consequently so is $n(x,t)$.

The steering's effect of the noise "sample" at the k:th hydrophone shall be characterized by the relationship

$$n_k(t) = \nu_k[t + (k-M-\tfrac{1}{2})T], \; k = 1,2,\ldots,2M \tag{A-1}$$

A.1 AN ASSUMPTION

If $R(\tau)$ is the normalized autocorrelation function of the noise process before steering, let us assume that we may combine time and space correlation along the array such that the covariance matrix of the noise after steering is given by

$$\begin{aligned} E[n_k(t)n_m(t)] &= E[\nu_k[t + (k-M-\tfrac{1}{2})T]\nu_m[t + (m-M-\tfrac{1}{2})T] \\ &= R_{km}[(m-k)T] = r_{km}s_k s_m R[(m-k)T], \end{aligned} \tag{A-2}$$

where the $R_{km}(\tau)$ are the interelement crosscorrelation functions, the $\{r_{km}\}$ are the "spatial correlation" coefficients, and the $\{s_k^2\}$ are the noise variances at the individual hydrophones.

In other words, we are supposing that the effects of steering on the noise process covariances can be modeled by the "cascading" of spatial effects (present for no steering, or T=0) and time delay effects, represented by the autocorrelation function of the process.

A.1.1 IMPLICATIONS FOR THE GENERAL CASE

The covariances of the multiplicative array sums are given by

$$E(n_1^2) = \sigma_1^2 = \sum_{k=1}^{M} \sum_{m=1}^{M} r_{km} s_k s_m R](m-k)T] \tag{A-3}$$

(the sum of the 'upper left quarter' of the array noise covariance matrix),

$$E(n_1 n_2) = \rho\sigma_1\sigma_2 = \sum_{k=1}^{M} \sum_{m=M+1}^{2M} r_{km} s_k s_m R[(m-k)T] \tag{A-4}$$

(the sum of the 'upper right quarter' or 'lower left quarter' of the array noise covariance matrix), and

$$E(n_2^2) = \sigma_2^2 = \sum_{k=M+1}^{2M} \sum_{m=M+1}^{2M} r_{km} s_k s_m R[(m-k)T] \tag{A-5}$$

A.1.2 IMPLICATIONS FOR THE NARROWBAND CASE

In the narrowband case we write $\nu_k(t) = \nu_{ck}(t) \cos \omega t + \nu_{sk}(t) \sin \omega t$. Davenport and Root [12] show that we may write

$$E[\nu_{ck}(t)\nu_{ck}(t+\tau)] = E[\nu_{sk}(t)\nu_{sk}(t+\tau)] = s_k^2 \int_{-\infty}^{\infty} df W_\nu(f) \cos 2\pi f\tau$$

$$= s_k^2 R_c(\tau), \text{ and} \tag{A-6}$$

$$E[\nu_{ck}(t)\nu_{sk}(t+\tau)] = -E[\nu_{ck}(t+\tau)\nu_{sk}(t)] = s_k^2 \int_{-\infty}^{\infty} df W_\nu(f) \sin 2\pi f\tau$$

$$= s_k^2 R_{cs}(\tau), \qquad \text{(A-7)}$$

where $W_\nu(f)$ is the normalized noise process spectrum. Thus $R_c(0) = 1$ and $R_{cs}(0) = 0$.

Using the same approach as in the preceding section, we write for the array sums

$$E(n_{1c}^2) = E(n_{1s}^2) = \sigma_1^2 = \sum_{k=1}^{M} \sum_{m=1}^{M} r_{km} s_k s_m R_c[(m-k)T] \qquad \text{(A-8)}$$

$$E(n_{1c} n_{1s}) = E(n_{2c} n_{2s}) = 0 \qquad \text{(A-9)}$$

$$E(n_{1c} n_{2c}) = E(n_{1s} n_{2s}) = \rho\sigma_1\sigma_2 = \sum_{k=1}^{M} \sum_{m=M+1}^{2M} r_{km} s_k s_m R_c[(m-k)T] \qquad \text{(A-10)}$$

$$E(n_{1c} n_{2s}) = -E(n_{1s} n_{2c}) = r\sigma_1\sigma_2 = \sum_{k=1}^{M} \sum_{m=M+1}^{2M} r_{km} s_k s_m R_{cs}[(m-k)T] \qquad \text{(A-11)}$$

$$E(n_{2c}^2) = E(n_{2s}^2) = \sigma_2^2 = \sum_{k=M+1}^{2M} \sum_{m=M+1}^{2M} r_{km} s_k s_m R_c[(m-k)T] \qquad \text{(A-12)}$$

A.2 EFFECT OF NOISE PROPERTIES ON ARRAY SUM COVARIANCES

The following is simply a compilation of certain noise properties, so far not specified, and their consequences for the array sum noise parameters we have just related to the input process: σ_1, σ_2, ρ, r.

Narrowband noise with a spectrum even about the center frequency

For this case $R_{cs}(\tau) = 0$, and $R(\tau) = R_c(\tau) \cos \omega\tau$, so that $r=0$.

Uniform variance across array, correlation proportional to distance

Here $s_k = s$, all k, and $r_{km} = f(|m-k|)$, so that $\sigma_1 = \sigma_2$.

White noise

We have $R_{cs}(\tau) = 0$, $R_c(\tau) = \delta(\tau)$, so that $\rho = r = 0$.

Spatially independent noise

If $r_{km} = \delta_{km}$, then $\rho = r = 0$.

(Note the duality between spatial properties and spectral properties.)

A.3 FURTHER MODELING OF THE SYMMETRIC NOISE COVARIANCE MATRIX

When the noise across the array is symmetric and either white noise or spatially independent, we have $\sigma_1^2 = \sigma_2^2 = \sigma^2 = Ms^2$ and $\rho = 0$. But we should like to be a bit more general than that. The following artifices are developed to provide an alternative to the white noise assumption.

A.3.1 ADJACENT CORRELATION ONLY

If we say that only the adjacent noise inputs are correlated with one another, with correlation coefficient a, it is easy to perform the summations of A.1.1 to find

$$\sigma^2 = Ms^2 + 2(M-1)as^2, \; \rho\sigma^2 = as^2, \; a<1. \qquad \text{(A-13)}$$

A.3.2 EXPONENTIALLY DECREASING CORRELATION

If we say that the correlation coefficient between hydrophones m and n is an exponentially decreasing function of the distance between them, that is $r_{mn}R[(m-n)T] = e^{-b|m-n|}$, we have probably a more useful representation. For small numbers of hydrophones the calculation is simple. For examples:

$$M = 1: \quad \sigma^2 = s \; , \quad = e^{-b}$$

$$M = 2: \quad \sigma^2 = 2s^2(1 + e^{-b}), \; \rho\sigma^2 = s^2(e^{-b} + 2e^{-2b} + e^{-3b})$$

$$M = 3: \quad \sigma^2 = s^2(3 + 4e^{-b} + 2e^{-2b}),$$

$$\rho\sigma^2 = s^2(e^{-b} + 2e^{-2b} + 3e^{-3b} + 2e^{-4b} + e^{-5b}).$$

It is apparent that for large M or for a general expression, unless terms are discarded (effectively what is done in the tridiagonal case) then our effort to achieve a closed expression is defeated. A way around this obstacle is to model the sums of offdiagonal terms as integrals, asserting that the sum is proportional to area of an appropriately chosen region under the curve $\exp(-b|x-y|)$. In this fashion we have

$$\sigma^2 = Ms^2 + 2M(M-1)s^2 \int_0^1 dx \int_0^x dy \; e^{-b(x-y)} = Ms^2 + 2M(M-1)s^2 \left(\frac{b+e^{-b}-1}{b^2}\right) \qquad \text{(A-14)}$$

$$\rho\sigma^2 = M^2s^2 \int_1^2 dx \int_0^1 dy \; e^{-b(x-y)} = M^2s^2e^{-b}\left(\frac{1-e^{-b}}{b}\right)^2 . \qquad \text{(A-15)}$$

Thus for b going to infinity, we have $\sigma^2 = Ms^2$ and $\rho = 0$, while for b going to zero, we have $\sigma^2 = \rho\sigma^2 = M^2s^2$.

APPENDIX B

ANOTHER FORM FOR THE PROBABILITY DENSITY FUNCTION

The case we are considering is that in which the cross-correlation coefficient $r = 0$ and $\sigma_1 = \sigma_2$ ($\rightarrow R = 0$), so that the term X in (2-21) equals zero. Under these conditions, the sum and difference envelopes Z_1 and Z_2 are statistically independent, since we may factor their joint probability density function (2-14):

$$p_4(Z_1,Z_2) = \frac{Z_1}{\sigma_3^2} \exp\left(-\frac{Z_1^2+S_3^2}{2\sigma_3^2}\right) I_0\left(\frac{S_3Z_1}{\sigma_3^2}\right) \frac{Z_2}{\sigma_4^2} \exp\left(-\frac{Z_2^2+S_4^2}{2\sigma_4^2}\right) I_0\left(\frac{S_4Z_2}{\sigma_4^2}\right)$$

$$= \frac{2Z_1}{\sigma_3^2} \quad \chi^2\left[\frac{Z_1^2}{\sigma_3^2};\ 2, 2h_3^2\right] \frac{2Z_2}{\sigma_4^2} \quad \chi^2\left[\frac{Z_2^2}{\sigma_4^2};\ 2, 2h_4^2\right], \qquad \text{(B-1)}$$

where $\sigma_3^2 = \frac{1}{2}\sigma^2(1+\rho)$, $\sigma_4^2 = \frac{1}{2}\sigma^2(1-\rho)$, $h_3^2 = S_3^2/2\sigma_3^2 = (h_1^2+h_2^2+2h_1h_2 \cos 2\theta)/2(1+\rho)$, and $h_4^2 = S_4^2/2\sigma_4^2 = (h_1^2+h_2^2-2h_1h_2 \cos 2\theta)/2(1-\rho)$. $\chi^2(x;2,a^2)$ is the noncentral chi-square distribution with two degrees of freedom and noncentrality parameter a^2. Thus the filter output $y = \frac{1}{2}(Z_1^2-Z_2^2)$ is the difference between two scaled, independent noncentral chi-square random variables. This type of relationship was noted also by Lee [16] in a similar application.

The noncentral chi-square distribution has many forms and interpretations (see [17-20]). For example, we may write

$$\chi^2(x;\ 2,\ a^2) = \exp(-\tfrac{1}{2}a^2) \sum_{r=0}^{\infty} \frac{(\frac{1}{2}a^2)^r}{r!} \chi^2(x;\ 2+2r), \qquad \text{(B-2)}$$

where $\chi^2(x;\ n)$ is the (central) chi-square distribution with n degrees of freedom:

$$\chi^2(x;\ n) = \frac{1}{2} e^{-\frac{1}{2}x} (x/2)^{\frac{1}{2}n-1} [\Gamma(n/2)]^{-1} \qquad \text{(B-3)}$$

B.1 Density for y using characteristic function

Using an expression given by Jayachandran and Barr [21], we may write the characteristic function of the distribution of y as

$$\phi_y(t) = \exp\ [-(h_3^2+h_4^2)] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(h_3^2)^m}{m!} \frac{(h_4^2)^n}{n!}$$

$$\times \left\{ \left(\frac{1+\rho}{2}\right)^{n+1} \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{[\frac{1}{2}(1-\rho)]^{m-k}}{(1-i\sigma_3^2 t)^{k+1}} + \left(\frac{1-\rho}{2}\right)^{m+1} \sum_{r=0}^{n} \binom{m+n-r}{m} \times \frac{[\frac{1}{2}(1+\rho)]^{n-r}}{(1+i\sigma_4^2 t)^{r+1}} \right\} \quad \text{(B-4)}$$

Thus we obtain for the density of y

$$p_6(y) = \frac{2}{\sigma^2} \exp\ [-(h_3^2+h_4^2)] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[\frac{1}{2}(1-\rho)h_3^2]^m}{m!} \frac{[\frac{1}{2}(1+\rho)h_4^2]^n}{n!}$$

$$\times \begin{cases} \sum_{k=0}^{m} \binom{m+n-k}{n} \left(\frac{1-\rho}{2}\right)^{-k} \chi^2\left[\frac{4y}{\sigma^2(1+\rho)}\ ;\ 2+2k\right], & y > 0 \\ \sum_{r=0}^{n} \binom{m+n-r}{m} \left(\frac{1+\rho}{2}\right)^{-r} \chi^2\left[\frac{-4y}{\sigma^2(1-\rho)}\ ;\ 2+2r\right], & y < 0 \end{cases} \quad \text{(B-5)}$$

Defining the polynomials $G_m^n(x) = \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k}{k!}$, we obtain finally

$$p_6(y) = \frac{1}{\sigma^2} \exp\ [-(h_3^2+h_4^2)] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{[\frac{1}{2}(1-\rho)h_3^2]^m}{m!} \frac{[\frac{1}{2}(1+\rho)h_4^2]^n}{n!}$$

$$\times \begin{cases} \exp\left[\frac{-2y}{\sigma^2(1+\rho)}\right] G_m^n \left[\frac{4y}{\sigma^2(1-\rho^2)}\right], & y \geq 0 \\ \exp\left[\frac{2y}{\sigma^2(1-\rho)}\right] G_n^m \left[\frac{-4y}{\sigma^2(1-\rho^2)}\right], & y < 0 \end{cases} \quad \text{(B-6)}$$

B.2 The polynomials $G_m^n(x)$

We list here several of the properties of these polynomials.

Definition: $G_m^n(x) \triangleq \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k}{k!}$; $G_m^0(x) = e_m(x)$, $G_0^n = 1$. (B-7)

Iterative relationships: $G_m^n(x) = \frac{x^m}{m!} + \sum_{r=0}^{n} G_{m-1}^r(x) = G_{m-1}^n(x) + G_m^{n-1}(x)$ (B-8)

also, $G_m^n(x) = \sum_{\substack{k=0 \\ (m,n>p)}}^{p} \binom{p}{k} G_{m-k}^{n-p+k}(x) = \sum_{\substack{k=0 \\ (n\geq m)}}^{m} \binom{m}{k} G_k^{n-k}(x) = \sum_{\substack{k=0 \\ (m\geq n)}}^{n} \binom{n}{k} G_{m-k}^{k}(x)$ (B-9)

Differentiation formula: $\frac{d^k}{dx^k} G_m^n(x) = G_{m-k}^n(x)$ (B-10)

Addition formula: $G_m^n(x+y) = \sum_{k=0}^{m} \frac{y^k}{k!} G_{m-k}^n(x)$ (B-11)

Relationships with special functions:

The polynomials $G_m^n(x)$ may be related to the confluent hypergeometric functions $\Psi(\alpha, \beta; x)$ and ${}_2F_0(\gamma, \delta; ; -1/x)$, to the Laguerre polynomials $L_m^\nu(x)$, and to the Whittaker functions $W_{\lambda,\mu}(x)$ by the following formulae.

$$G_m^n(x) = \frac{1}{m!} \Psi(-m, -m-n; x) \tag{B-12}$$

$$= \frac{x^{m+n+1}}{m!} \Psi(n+1, m+n+2; x) \tag{B-13}$$

$$= \frac{x^m}{m!} {}_2F_0(-m, n+1; ; -1/x) \tag{B-14}$$

$$= \frac{1}{m!} e^{x/2} x^a W_{b,a+\frac{1}{2}}(x) \text{ for } a = \frac{m+n}{2},\ b = \frac{m-n}{2} \tag{B-15}$$

$$= (-1)^m L_m^{-m-n-1}(x). \tag{B-16}$$

By using the identity (B-15), our expression (B-6) for the pdf can be shown to be identical to that in [63, p. 63, eq. 28].

B.3 Equivalence of forms

We now show that the form we have just derived for the probability density function is equivalent to that of (2-23) under the assumed conditions.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^m b^n}{m!n!} G_m^n(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k a^m b^n}{k!m!n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{m+n}{n} \frac{x^k a^{m+k} b^n}{k!(m+k)!n!}$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{a^{m+k} x^k}{(m+k)!k!} \, {}_1F_1(m+1,1;\, b)$$

$$= \sum_{m=0}^{\infty} \left(\frac{a}{x}\right)^{\frac{1}{2}m} I_m(2\sqrt{ax}) \, {}_1F_1(m+1,1;\, b)$$

$$= e^b \sum_{m=0}^{\infty} \left(\frac{a}{x}\right)^{\frac{1}{2}m} I_m(2\sqrt{ax}) \, L_m(-b), \qquad \text{(B-17)}$$

using Kummer's transformation on the last step.

APPENDIX C

INVERSION OF THE DIRECTIVITY FUNCTION

The directivity function, as it has been defined in Chapter 1, is

$D = D(\phi; N) = \sin N\phi / N\sin\phi$

$$= \frac{2}{N}[\cos\phi + \cos 3\phi + \cos 5\phi + \ldots + \cos(N-1)\phi], \text{ N even} \qquad \text{(C-1)}$$

$$= \frac{1}{N}[1 + 2\cos 2\phi + 2\cos 4\phi + \ldots + 2\cos(N-1)\phi], \text{ N odd.} \qquad \text{(C-2)}$$

In this appendix we derive an inversion of D, that is, in a way to express $\phi = \phi(D; N)$.

Recalling that $\cos nx = 1 - \frac{(nx)^2}{2} + \frac{(nx)^4}{24} - \ldots + \ldots$

$$= 1 - (nx)^2/2, \ (nx)^2 << 12, \qquad \text{(C-3)}$$

we may write for D, N even,

$$D = \frac{2}{N}[1 - \phi/2 + 1 - 9\phi^2/2 + 1 - 25\phi^2/2 + \ldots + 1 - (N-1)^2\phi^2/2]$$

$$= \frac{2}{N}\left\{\frac{N}{2} - \frac{\phi^2}{2}[1^2 + 3^2 + 5^2 + \ldots + (N-1)^2]\right\}$$

$$= 1 - (N^2-1)\phi^2/6, \quad [(N-1)\phi]^2 << 12 \qquad \text{(C-4)}$$

where we have used Gradshteyn and Ryzhik [13], formula 0.122.2. For N odd, we obtain

$$D = \frac{1}{N}[1 + 2 - 4\phi^2 + 2 - 16\phi^2 + 2 - 36\phi^2 + \ldots + 2 - (N-1)^2\phi^2]$$

$$= \frac{1}{N}\left\{N - 4\phi^2 [1^2 + 2^2 + 3^2 + \ldots + (N-1)^2/4]\right\}$$

$$= 1 - (N^2-1)\phi^2/6 \qquad \text{(C-5)}$$

where once again we have used [13], formula 0.121.2. Therefore, for N odd or even we can write

$$\phi^2 = 6(1-D)/(N^2-1). \tag{C-6}$$

If we use the first three terms of the cosine power series representation, we obtain

$$D = 1 - (N^2-1)\phi^2/6 + \frac{4}{12N}[1^4 + 3^4 + 5^4 + \ldots + (N-1)^4]$$

$$= 1 - (N^2-1)\phi^2/6 + (3N^4-10N^2+7)\phi^4/360, \quad [(N-1)\phi]^2 << 30. \tag{C-7}$$

Application to Chapter 4.

In Chapter 4, we use $\phi = \beta\sin\alpha/2$. Therefore we have the approximation

$$\beta^2\sin^2\alpha/4 = 6(1-D)/(N^2-1)$$

$$= \beta^2(1-\cos2\alpha)/2, \tag{C-8}$$

$$\text{or} \quad \alpha = \arcsin\frac{2}{\beta}\sqrt{\frac{6(1-D)}{N^2-1}} = \frac{1}{2}\arccos\left\{1 - \frac{48(1-D)}{\beta^2(N^2-1)}\right\}, \quad \left(\frac{N-1}{2}\beta\sin\alpha\right)^2 << 12. \tag{C-9}$$

APPENDIX D

COMPUTER PROGRAMS

It is not our purpose here to burden the reader with a full disclosure of the many numerical procedures we have employed. We shall list, however, for the benefit of those who can interpret them, the major computer programs we have written, so that they may be open to scrutiny and available to those who may wish to perform similar calculations.

The programs are all straightforwardly written in the BASIC language. Those programs listed are labeled as follows:

(1) basic multiplicative probability density function (2-26)

(2) basic noncentral chi-square probability density function (2-25)

(3) multiplicative array detection analysis (3-2, 3-10)

(4) standard array detection analysis (3-5, 3-13)

(5) multiplicative array estimator mean and mean square (4-69, 4-70)

(6) standard array estimator mean and mean square (4-53, 4-54)

(7) multiplicative array estimator bound (5-48)

(8) standard array estimator bound (5-38).

Some manual calculations were employed to supplement programs 5 and 7, using [31].

PROGRAM ONE

```
1     REM BASIC PRØGRAM FØR P6(Y)
5     DIM G(100,100),H(100),J(100),U(100)
10    LET G(0,0)=1
15    LET H(0)=1
20    LET J(0)=1
25    LET U(0)=1
98    REM LINES 100 TØ 335: ENTER RHØ, SIGMA 2, PØWER SNR, AND THETA;
99    REM                       CØMPUTE CØNSTANTS.
100   READ R,S,H,T
200   PRINT "              PRØBABILITY DENSITY FUNCTIØN"
205   PRINT
210   PRINT "RHØ="R, "SIG2="S, "PSNR="H,"THETA="T
215   PRINT
220   PRINT "Y","P6(Y)"
305   LET C1=2*H*(1-R*CØS(2*T))/(1-R↑2)
310   LET C2=H*(CØS(T))↑2
315   LET C3=H*(SIN(T))↑2
320   LET C4=(1-R)/(1+R)
325   LET C5=-2/(S*(1+R))
330   LET C6=2/(S*(1-R))
335   LET C7=4/(S*(1-R↑2))
400   READ A1,A2,A3,L
405   FØR Y=A1 TØ A2 STEP A3
410   LET Y1=C7*ABS(Y)
415   LET P=0
498   REM LINES 500 TØ 545: CALCULATE THE PØLYNØMIALS G(Y;K,N).
500   FØR N=1 TØ L
505   LET G(0,N)=1
510   LET H(N)=C3*H(N-1)/N
515   FØR K=1 TØ L
520   LET U(K)=Y1*U(K-1)/K
525   LET G(K,0)=G(K-1,0) + U(K)
530   LET G(K,N)=G(K-1,N) + G(K,N-1)
540   LET J(K)=C2*J(K-1)/K
542   NEXT K
545   NEXT N
599   REM LINES 600 TØ 725: CØMPUTE DENSITY FUNCTIØN.
600   IF Y<0 THEN 700
605   FØR N=0 TØ L
610   FØR K=0 TØ L
615   LET P=P +(C4↑(K-N))*J(K)*H(N)*G(K,N)
620   NEXT K
625   NEXT N
630   LET P=P*EXP(C5*Y-C1)/S
635   GØ TØ 800
700   FØR N=0 TØ L
705   FØR K=0 TØ L
710   LET P=P + (C4↑(K-N))*J(K)*H(N)*G(N,K)
715   NEXT K
720   NEXT N
725   LET P=P*EXP(C6*Y-C1)/S
```

```
800   PRINT Y,P
805   NEXT Y
1020  DATA .2,1,1,0
1040  DATA -7.5,7.5,.5
1041  DATA 30
2000  END
```

PROGRAM TWO

```
5     READ A1,A2,A3
10    READ H,S
15    PRINT "Y","P(Y)"
20    FØR Y=A1 TØ A2 STEP A3
25    LET U=2*SQR(H*Y/S)
30    GØSUB 300
35    LET P=K*EXP(-H-Y/S)/S
40    PRINT Y,P
45    NEXT Y
50    DATA 0,10,1
55    DATA 1,1
60    STØP
300   LET V=.5*U
305   LET K=1
310   LET G=1
315   FØR N=1 TØ 20
320   LET G=G*V↑2/N↑2
325   LET K=K + G
330   NEXT N
335   RETURN
500   END
```

PROGRAM THREE

```
5     DIM F(100),H(100),E(100),U(100),V(100)
10    LET H(0)=1
15    LET U(0)=1
20    LET V(0)=1
30    LET F(0)=1
32    READ M,B,A,H1,S1,A4
35    LET E=.5*M*B*SIN(A)
40    LET D=SIN(E)/(M*SIN(E/M))
45    LET H=H1*(M*D)↑2/(M + 2*(M-1)*A4)
50    LET R=A4/(M + 2*(M-1)*A4)
55    LET S=M*S1 + 2*(M-1)*A4*S1
60    LET T=E
65    PRINT "          MULTIPLICATIVE ARRAY DETECTION ANALYSIS"
70    PRINT
75    PRINT "M="M;"BETA="B;"ALFA="A; "CORREL="A4
80    PRINT "INPUT SNR="H1;"S1="S1
85    PRINT "RHØ="R;"SIG↑2="S;"SNR="H;"THETA="T
90    PRINT
95    READ L
105   READ A1,A2,A3
110   PRINT "Y","PD","PF"
```

```
115   LET C1=H*(CØS(T))↑2
120   LET C2=H*(SIN(T))↑2
125   LET C3=(1-R)/(1+R)
130   LET C4=2/(1-R)
135   FØR Y=A1 TØ A2 STEP A3
140   LET C5=2*Y/(S*(1+R))
145   LET P1=0
150   FØR N=0 TØ 1
155   LET U(N+1)=C2*U(N)/(N+1)
160   FØR M=0 TØ L
165   LET V(M+1)=C1*V(M)/(M+1)
170   FØR K=0 TØ M
175   LET F(K+1)=F(K)*(N+K+1)/(K+1)
180   LET E(M-K)=0
185   FØR J=0 TØ M-K
190   LET H(J+1)=C5*H(J)/(J+1)
195   LET E(M-K)=E(M-K) + H(J)
200   NEXT J
205   LET P2=U(N)*V(M)*(C3↑(M-N))
210   LET P3=F(K)*(C4↑(M-K))*E(M-K)
215   LET P1=P1 + P2*P3
220   NEXT K
225   NEXT M
230   NEXT N
235   LET P4=-C5-2*H*(1-R*CØS(2*T))/(1-R↑2)
240   LET P5=.5*(1+R)*EXP(P4)*P1
245   LET P6=.5*(1+R)*EXP(-C5)
250   PRINT Y,P5,P6
255   NEXT Y
260   DATA 5,3.14159,.00001
265   DATA 1,1,0
270   DATA 30
275   DATA 0,70,5
280   END
```

PROGRAM FOUR

```
5     DIM U(200),H(200),E(200)
10    LET U(0)=1
15    LET H(0)=1
20    READ N,B1,A5,S1,H1,A4,L,K2,L1
25    PRINT "               STANDARD ARRAY DETECTIØN ANALYSIS"
30    PRINT
35    LET T=.5*N*B1*SIN(A5)
40    LET D=SIN(T)/(N*SIN(T/N))
45    LET S=N*S1 + 2*(N-1)*A4*S1
50    LET H=H1*(N*D)↑2/(N + 2*(N-1)*A4)
55    PRINT "N="N;"BETA="B1;"ALPHA="A5
60    PRINT "S1="S1,"H1="H1,"A="A4
65    PRINT "S="S,"H="H,"THETA="T
66    PRINT "L="L,"K2="K2,"L1="L1
67    PRINT
68    PRINT "Y","PD","PF"
69    READ A1,A2,A3
```

```
70   FØR Y2=A1 TØ A2 STEP A3
75   LET Y1=Y2/S
80   LET A=SQR(2*H)
85   LET B=SQR(2*Y1)
140  IF (A+B)<20 THEN 200
145  LET K1=(A+B-20)/(K2+1)
150  LET A=A-K1
155  LET B=B-K1*K2
200  IF A>B THEN 220
205  LET X=.5*A↑2
210  LET Y=.5*B↑2
215  GØ TØ 300
220  LET X=.5*B↑2
225  LET Y=.5*A↑2
300  LET Q1=0
305  FØR M=0 TØ L
310  LET U(M+1)=Y*U(M)/(M+1)
315  LET E(M)=0
320  FØR K=0 TØ M
325  LET H(K+1)=X*H(K)/(K+1)
330  LET E(M)=E(M) + H(K)
335  NEXT K
340  LET Q1=Q1 + U(M)*E(M)
345  NEXT M
350  LET Q2=Q1*EXP(-X-Y)
405  IF A<B THEN 416
410  LET Q=Q2
415  GØ TØ 425
416  GØSUB 550
420  LET Q=1 - Q2 + K*EXP(-X-Y)
425  LET P1=EXP(-Y1)
430  PRINT Y2,Q,P1
431  NEXT Y2
480  DATA 4,3.14159,.00001
481  DATA 1,1,0
498  DATA 120,.93,30
500  DATA 0,50,5
545  STØP
550  LET V=.5*A*B
555  LET K=1
560  LET G=1
565  FØR N=1 TØ L1
570  LET G=G*V↑2/N↑2
575  LET K=K + G
580  NEXT N
585  RETURN
600  END
```

PROGRAM FIVE

```
5    DIM U(94,94),L(94,94),B(94),C(94)
10   LET U(0,0)=1
15   LET L(0,0)=1
20   LET B(0)=4/15
```

```
25    LET B(1)=32/315
30    LET C(0)=1/6
35    LET C(1)=1/20
40    READ M,B,H1,N2
50    FØR R=1 TØ N2
55    LET B(R+1)=2*(R+1)*(2*R+3)*B(R)/((2*R+3.5)*(2*R+4.5))
60    LET C(R+1)=(R+1)*(2*R+3)*C(R)/((R+2)*(2*R+5))
65    NEXT R
100   LET K=2*SQR(6/(4*M↑2-1))/B
105   PRINT K
110   LET K1=M*H1
115   FØR A=.00001 TØ .21 STEP .01
120   LET E=.5*M*B*SIN(A)
125   LET D=SIN(E)/(M*SIN(E/M))
130   LET L=SQR(1+4*K1*D↑2)
135   LET G=.25 + 2*K1*D↑2*(SIN(E))↑2
140   LET G1=SQR(G)
145   LET A9=G*L↑2
150   LET B9=-2*K1
155   LET C9=.5*L↑2
160   LET L(1,0)=1-C9
165   LET L(2,0)=1-2*C9+.5*C9↑2
170   FØR R=0 TØ N2
180   LET U(R+1,0)=B9*U(R,0)/(R+1)
190   LET U(R,1)=A9*U(R,0)/(R+1)
200   FØR S=1 TØ N2
210   LET U(R+1,S)=B9*U(R,S)/(R+S+1)
220   LET U(R,S+1)=A9*U(R,S)/((R+S+1)*(S+1))
230   LET L(S+1,R)=((2*S+R+1-C9)*L(S,R)-(R+S)*L(S-1,R))/(S+1)
240   NEXT S
250   LET L(0,R+1)=((R+1)*L(0,R)-L(1,R))/C9
260   LET L(1,R+1)=((R+2)*L(1,R)-2*L(2,R))/C9
270   NEXT R
300   LET M1=0
305   LET M2=0
310   FØR R=0 TØ N2
315   FØR S=0 TØ N2
320   LET M1=M1 + U(R,S)*L(R,S)*B(R)
325   LET M2=M2 + U(R,S)*L(R,S)*C(R)
330   NEXT S
335   NEXT R
340   LET M1=M1*4*K*K1*EXP(-C9-2*G)
345   LET M2=M2*4*K↑2*K1*EXP(-C9-2*G)
350   PRINT A,M1,M2,L,2*G1
355   NEXT A
400   DATA 5,3.14159,4,93
415   END
```

PROGRAM SIX

```
5     DIM U(100),B(100),L(100),C(100)
10    LET U(0)=1
15    LET B(0)=4/15
20    LET B(1)=32/315
```

```
25    LET L(0)=1
30    LET C(0)=1/6
35    LET C(1)=1/20
40    READ N,B,H1,N2
45    LET K=2*SQR(6/(N↑2-1))/B
50    LET K1=N*H1
52    LET U(1)=-K1
55    FØR A=.00001 TØ .21 STEP .01
60    LET E=.5*N*B*SIN(A)
65    LET D=SIN(E)/(N*SIN(E/N))
70    LET H=N*H1*D↑2
75    LET L(1)=1-H
80    LET M1=B(0)
85    LET M2=C(0)
90    FØR N1=1 TØ N2
100   LET U(N1+1)=-K1*U(N1)/(N1+1)
102   LET B(N1+1)=2*(N1+1)*(2*N1+3)*B(N1)/((2*N1+3.5)*(2*N1+4.5))
105   LET C(N1+1)=(N1+1)*(2*N1+3)*C(N1)/((N1+2)*(2*N1+5))
110   LET L(N1+1)=((2*N1+1-H)*L(N1)-N1*L(N1-1))/(N1+1)
115   LET M1=M1 + U(N1)*L(N1)*B(N1)
120   LET M2=M2 + U(N1)*L(N1)*C(N1)
121   NEXT N1
122   LET M1=2*M1*K*K1*EXP(-H)
123   LET M2=M2*2*K↑2*K1*EXP(-H)
125   LET V1=M2-M1↑2
130   LET V2=M2-2*A*M1+A↑2
135   PRINT A,M1,M2,V1,V2
140   NEXT A
145   DATA 10,3.14159,4
150   DATA 99
200   END
```

PROGRAM SEVEN

```
5     DIM U(94,94),L(94,94),B(94)
10    LET U(0,0)=1
15    LET L(0,0)=1
20    LET B(0)=4/15
25    LET B(1)=32/315
30    READ M,B,H1,N2
40    FØR R=1 TØ N2
50    LET B(R+1)=2*(R+1)*(2*R+3)*B(R)/((2*R+3.5)*(2*R+4.5))
65    NEXT R
100   LET K=2*SQR(6/(4*M↑2-1))/B
105   PRINT K
110   LET K1=M*H1
115   FØR A=.00001 TØ .21 STEP .01
120   LET E=.5*M*B*SIN(A)
125   LET D=SIN(E)/(M*SIN(E/M))
130   LET L=SQR(1+4*K1*D↑2)
135   LET G=.25 + 2*K1*D↑2*(SIN(E))↑2
140   LET G1=SQR(G)
145   LET A9=G*L↑2
150   LET B9=-2*K1
155   LET C9=.5*L↑2
```

```
160   LET L(1,0)=2-C9
165   LET L(2,0)=3-3*C9+.5*C9↑2
200   FØR R=0 TØ N2
215   LET U(R+1,0)=B9*U(R,0)/(R+1)
220   LET U(R,1)=A9*U(R,0)/(R+1)
225   FØR S=1 TØ N2
230   LET U(R+1,S)=B9*U(R,S)/(R+S+1)
235   LET U(R,S+1)=A9*U(R,S)/((R+S+1)*(S+1))
240   LET L(S+1,R)=((2*S+R+2-C9)*L(S,R)-(R+S+1)*L(S-1,R))/(S+1)
245   NEXT S
246   LET L(0,R+1)=((R+2)*L(0,R)-L(1,R))/C9
247   LET L(1,R+1)=((R+3)*L(1,R)-2*L(2,R))/C9
250   NEXT R
300   LET M1=0
310   FØR R=0 TØ N2
315   FØR S=0 TØ N2
320   LET M1=M1 + U(R,S)*L(R,S)*B(R)*(R+S+1-2*G)/(R+S+1)
330   NEXT S
335   NEXT R
340   LET M1=M1↑2*K↑2*8*K1↑2*L↑2*EXP(-L↑2-4*G)/(L↑2-1)
350   PRINT A,M1,L,2*G1
355   NEXT A
400   DATA 5,3.14159,2,60
415   END
```

PROGRAM EIGHT

```
5     DIM U(100),B(100),L(100)
10    LET U(0)=1
15    LET B(0)=4/15
20    LET B(1)=32/315
25    LET L(0)=1
40    READ N,B,H1,N2
45    LET K=2*SQR(6/(N↑2-1))/B
50    LET K1=N*H1
52    LET U(1)=-K1
55    FØR A=.00001 TØ .21 STEP .01
60    LET E=.5*N*B*SIN(A)
65    LET D=SIN(E)/(N*SIN(E/N))
70    LET H=N*H1*D↑2
75    LET L(1)=2-H
80    LET M1=B(0)
90    FØR N1=1 TØ N2
95    LET U(N1+1)=-K1*U(N1)/(N1+1)
100   LET B(N1+1)=2*(N1+1)*(2*N1+3)*B(N1)/((2*N1+3.5)*(2*N1+4.5))
110   LET L(N1+1)=(2-H/(N1+1))*L(N1)-L(N1-1)
115   LET M1=M1 + U(N1)*L(N1)*B(N1)
121   NEXT N1
122   LET M1=2*M1*K*K1↑2*D*EXP(-H)
130   LET M1=M1↑2/(2*N*H1*(2*H-1))
135   PRINT A,M1
140   NEXT A
145   DATA 10,3.14159,.5,99
200   END
```

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SIGNAL DETECTION AND BEARING ESTIMATION BY SQUARE-LAW AND MULTIPLICATIVE ARRAY PROCESSORS

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ABSTRACT

Analysis is made of a "multiplicative" receiving array model which is known to have about one-half the beamwidth of the "additive" or conventional (square-law) array with the same number of elements. The probability density functions for the filter outputs of array models are shown for monochromatic, planewave signals in the presence of narrowband Gaussian noise, for broadside arrays with equal spacing and uniform gains. The number of elements, SNR, source bearing, and interelement noise correlation are treated as parameters of the distribution, and their influences are displayed graphically.

Based on these probability density functions for the square-law and the multiplicative array processors, we then examine their performances as signal detectors and as bearing estimators.

INTRODUCTION

In judging the capabilities of sonar array processing, it is a common practice to use the processor output SNR (signal-to-noise ratio) as a figure of merit for detection, and the array configuration's beamwidth as a measure of bearing estimation performance. Thus, based on these criteria Arndt[1], for example, concludes that conventional or additive processing is the best method for detection purposes, while correlating or multiplicative processing is the choice for obtaining bearing. These conclusions had also been reached by Welsby and Tucker [2] with some reservations. However, we wish to point out that these criteria, SNR and beamwidth, while often adequate for pursuing the optimization of a given configuration [3], are not always suitable for motivating a choice between two different configurations.

The usual standard of detection performance is the probability of detection (PD) achieved for a given false alarm rate (PFA) at an assumed system input SNR (Neyman-Pearson model of detection). Therefore, even though it is reasonable to assume that the PD is proportional to output SNR for each of two different systems, in general one cannot say that the PD's are comparable if the SNR is the same since there is yet another parameter to be specified: the detection threshold, or equivalently the the false alarm rate on which the threshold is based.

Similarly, while beamwith (resolution) is certainly a factor, bearing estimation performance is to be understood in terms of minimum error attainable from an arrangement in which the bearing is extracted, or indirectly measured, from observations corrupted by noise. Therefore, a true comparison of two configurations with respect to bearing estimation requires an accounting of the efficiency with which each smooths the noise.

The limitations we have pointed out are, of course, more or less understood by those employing the conventional figures of merit. The limitations are tolerated because of the difficulty in obtaining the precise quantities on which systems comparisons need to be based, quantities which are related to the statistical behavior of the system outputs. In this paper we present the probability density function of the detector output for a class of multiplicative arrays, and employ it to calculate detection and bearing estimation capabilities of these arrays in comparison with conventional or additive arrays which use a square-law detector. Though necessarily we select a narrow class of systems in order to achieve the rigor we seek, both the analytical approach and the results (described more fully in [4]) lend themselves to extensions, thereby encouraging more thorough system analysis in the field of sonar arrays.

MULTIPLICATIVE ARRAY PROBABILITY DENSITY FUNCTION

Consider the model of a multiplicative array as shown in Figure 1, in which the 2M omnidirectional receiving elements are equally spaced d units apart along a straight line. The sum of one half of the array element outputs is multiplied by that of the other half, and the product is lowpass filtered. We make the following assumptions:

SIGNAL: the array intersects a planewave, monochromatic signal with known amplitude A and unknown bearing α, relative to the perpendicular of the array.

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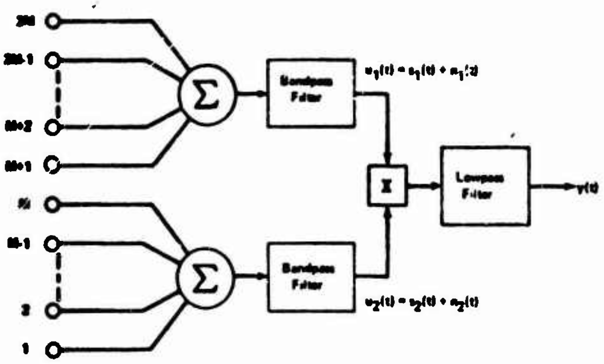


FIGURE 1. Multiplicative array model.

NOISE: the array elements experience stationary, zero-mean Gaussian noise such that the variance at each element is s^2 and, at a given instant of time, adjacent elements only are correlated, with coefficient a.

Under these conditions the sum channels of Figure 1 have the signal terms

$$s_i(t) = MAD_M\cos(\omega t-\theta_i), \qquad i = 1,2$$

where the directivity function is given by $D_M = \sin(\frac{1}{2}M\beta\sin\alpha)/M\sin(\frac{1}{2}\beta\sin\alpha)$, with $\beta = 2\pi d/\lambda$ and $2\theta = \theta_1-\theta_2 = M\beta\sin\alpha$. The noise terms $n_i(t)$ have equal variances $\sigma^2 = Ms^2 + 2(M-1)as^2$ and are correlated with coefficient $\rho = a/[M + 2(M-1)a]$.

The filter output is given by $y(t)=\frac{1}{2}(Z_1{}^2-Z_2{}^2)$ where Z_1 is the envelope of $\frac{1}{2}(u_1 + u_2)$ and Z_2 is the envelope of $\frac{1}{2}(u_1 - u_2)$, assuming ideal lowpass filtering. That is, y is the difference between two independent, scaled noncentral chi-square random variables (for n_i of unequal variance, the Z_i are dependent). It can be shown [4] that the probability density function of y in this case is

$$p_m(y) = p_\omega(y;\, \alpha,\, \beta,\, M,\, s,\, H,\, a)$$

$$= \exp\left\{\frac{-2y}{Ms^2(B+C)} - \frac{2MH^2D_M^2}{B}\,\frac{B-C\cos2\theta}{B^2-C^2} + \frac{MH^2D_M^2\sin^2\theta}{B}\,\frac{B+C}{B-C}\right\}$$

$$\times \frac{1}{BMs^2}\sum_{r=0}^{\infty}\left(\frac{MsHD_M\cos\theta}{2\sqrt{y}}\,\frac{B-C}{B}\right)^r \tag{1a}$$

$$\times I_r\left[\frac{4HD_M\cos\theta\sqrt{y}}{s(B+C)}\right] L_r\left[\frac{-MH^2D_M^2\sin^2\theta}{B}\,\frac{B+C}{B-C}\right],\quad y>0$$

$$= \exp\left\{\frac{2y}{Ms^2(B-C)} - \frac{2MH^2D_M^2}{B}\,\frac{B-C\cos2\theta}{B^2-C^2} + \frac{MH^2D_M^2\cos^2\theta}{B}\,\frac{B-C}{B+C}\right\}$$

$$\times \frac{1}{BMs^2}\sum_{r=0}^{\infty}\left(\frac{MsHD_M\sin\theta}{2\sqrt{-y}}\,\frac{B+C}{B}\right)^r \tag{1b}$$

$$\times I_r\left[\frac{4HD_M\sin\theta\sqrt{-y}}{s(B-C)}\right] L_r\left[\frac{-MH^2D_M^2\cos^2\theta}{B}\,\frac{B-C}{B+C}\right],\quad y<0$$

where $B = \sigma^2/Ms^2$, $C = \rho B$, and $H^2 = A^2/2s^2$, and where the $I_r(\ )$ are the modified Bessel functions of the first kind of order r, and the $L_r(\ )$ are the r:th degree Laguerre polynomials. The uncorrelated noise case corresponds to B = 1 and C = 0.

Results for the multiplicative array probability density function (1) are presented in Figures 2 through 5. The nominal case of 2M = 10 elements, uncorrelated noise (a=0), unit noise variance (s^2=1), zero bearing (α=0), half-wavelength array spacing ($\beta=\pi$), and zero dB input SNR (H^2=1) was chosen. In Figure 2 the SNR is varied, while the

FIGURE 2. Multiplicative array PDF (equation 1) for input signal-to-noise ratio varied.

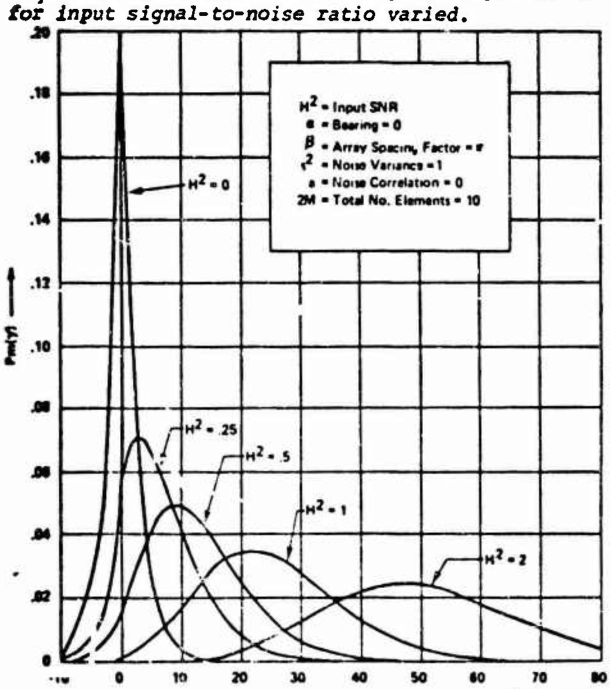
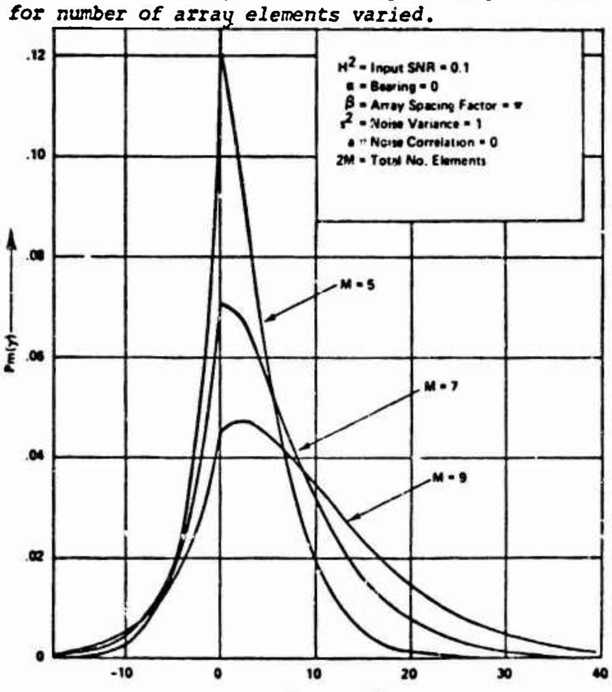


FIGURE 3. Multiplicative array PDF (equation 1) for number of array elements varied.



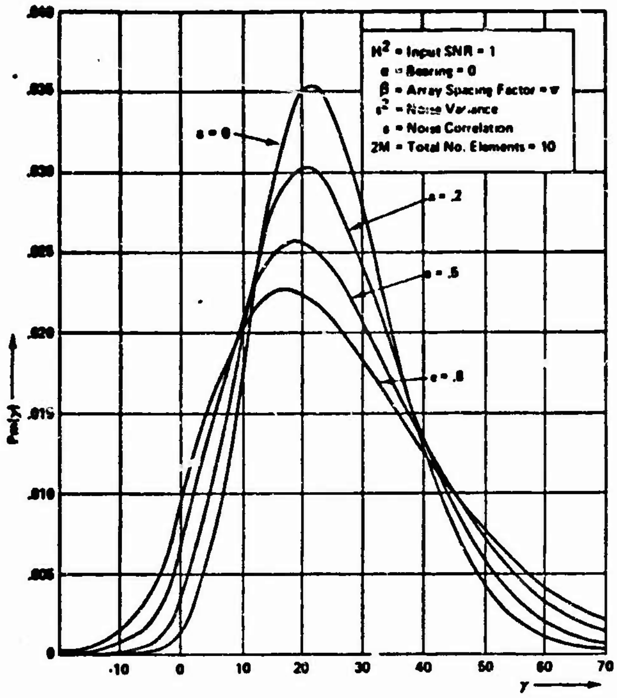
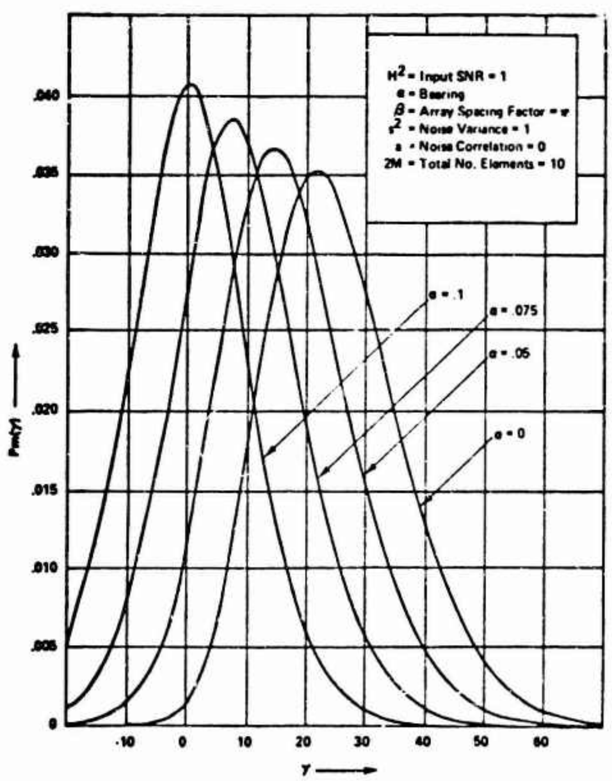


FIGURE 4. Multiplicative array PDF (equation 1) for noise correlation coefficient varied.

number of elements is varied in Figure 3. It is seen that these two parameters have similar effects on the distribution. The effect of adjacent interelement correlation is displayed in Figure 4,

FIGURE 5. Multiplicative array PDF (equation 1) for bearing angle varied.



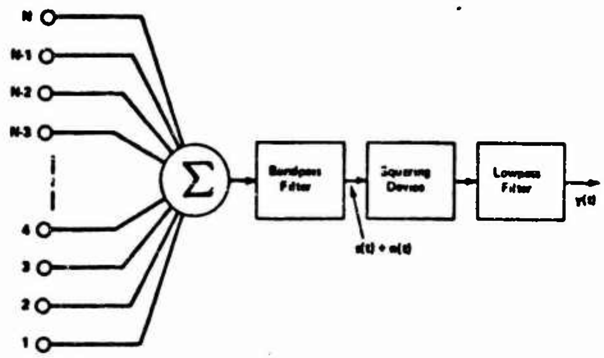


FIGURE 6. Square-law array model.

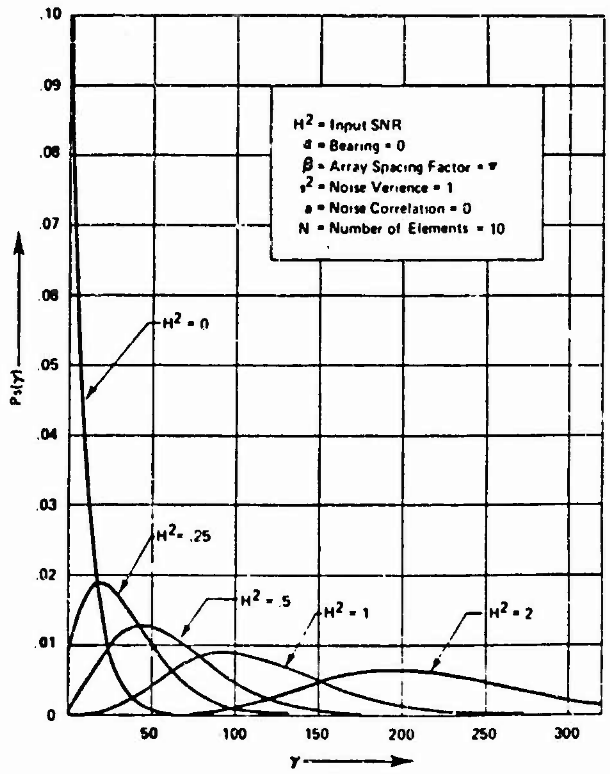
showing how increased correlation increases the variance of the filter output, and decreases the value of y at which the peak of the distribution occurs. Figure 5 demonstrates a phenomenon of interest, for the bearing is seen to act as a location parameter for the distribution, indicating the potential of using the filter output for bearing estimation.

SQUARE-LAW ARRAY PROBABILITY DENSITY FUNCTION

Consider the conventional or square-law array with N elements, as modelled in Figure 6. Using the same basic assumptions as above, it can be shown that the filter output y is a scaled noncentral chi-square random variable with probability density function

$$p_s(y) = p_s(y; \alpha, \beta, N, s, H, a)$$

FIGURE 7. Square-law array PDF (equation 2) for input signal-to-noise ratio varied.



$$\text{or } p_s(y) = \frac{1}{B'Ns^2}\exp\left[\frac{-(y+N^2H^2D_N^2s^2)}{B'Ns^2}\right] I_0\left[\frac{2HD_N\sqrt{y}}{B's}\right], \quad y>0$$
$$= 0, \quad y<0 \qquad (2)$$

where $B' = [N + 2(N-1)a]/N$

and $D_N = \sin(\frac{1}{2}N\beta\sin\alpha)/N\sin(\frac{1}{2}\beta\sin\alpha)$.

In Figures 7 and 8, the probability density function of the filter output of the square-law array model is seen to behave with parameter variation in much the same way as that of the multiplicative. Here we have used $H^2 = 1$, $s^2 = 1$, $\beta = \pi$, $\alpha = 0, a=0$, and $N = 10$ as the nominal case. In Figure 7 the SNR is varied, and in Figure 8 the source bearing is varied.

DETECTION PERFORMANCE

Having now the probability density functions for the array processor outputs, we are in a position to evaluate what Tucker [5] described as "...the difficult question of whether the better signal/noise performance...is really indicative of a higher probability of detection." That is, we are able directly to calculate the probabilities of detection for the two array models, as functions of SNR and of corresponding probabilities of false alarm.

Assuming the decision "signal present" is made if the filter output y exceeds a threshold τ, the probability of false alarm or false alarm rate and the probability of detection are given by

$$PFA = \int_\tau^\infty p(y|H=0)dy, \quad PD = \int_\tau^\infty p(y|H>0)dy. \qquad (3)$$

For the multiplicative array model we have $(\tau>0)$

$$PFA = \tfrac{1}{2}(1 + C/B)\exp\left[\frac{-2\tau}{Ms^2(B+C)}\right] \qquad (4)$$

$$PD = \frac{B+C}{2B}\exp\left[\frac{-2\tau}{Ms^2(B+C)} - \frac{2MH^2}{B}D_M^2\frac{B-C\cos 2\theta}{B^2-C^2}\right]$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{MH^2}{B}D_M^2\right)^{m+n}\frac{(\cos\theta)^{2m}}{m!}\frac{(\sin\theta)^{2n}}{n!} \qquad (5)$$
$$\times \left(\frac{B-C}{B+C}\right)^{m-n}\sum_{k=0}^{m}\binom{m+n-k}{n}\left(\frac{2B}{B-C}\right)^k e_k\left[\frac{2\tau}{Ms^2(B+C)}\right].$$

For the square-law array model we have

$$PFA = \exp(-\tau/B'Ns^2) \qquad (6)$$

and $PD = Q(HD_N\sqrt{2N/B'}, \sqrt{2\tau/B'Ns^2})$, (7)

where Q is Marcum's Q-function.

It can be shown that the multiplicative model has a more desirable false alarm rate for the same value of detection threshold, while the square-law model yields a higher PD. However, this apparent tradeoff is blurred by the unsuitability of the detection threshold value as a basis for comparison. The more informative "receiver operating characteristics" (ROC), given by PD = f(PFA; SNR), render two different threshold detectors more directly comparable, enabling a performance index such as "minimum detectable signal" (the SNR required to insure a given PD for a specified

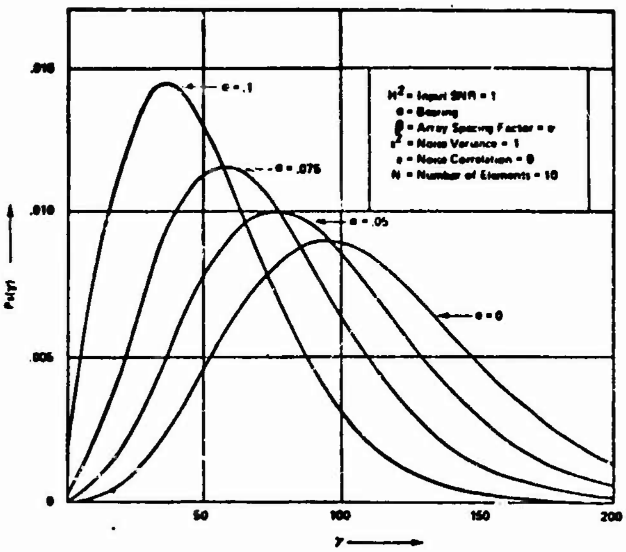


FIGURE 8. Square-law array PDF (equation 2) for the bearing angle varied.

maximum PFA) to be used to pronounce one the better detector with some confidence in the generality of this kind of statement. Helmstrom [6] cites additional uses of the ROC.

ROC curves for our two models, for the uncorrelated, boresight (α=0) case are given in Figure 9 for N=2M=4 and in Figure 10 for N=2M=10. The fact that the square-law array processor curves are consistently above those of the multiplicative indicates that the square-law array model is a better detector in this case. They are quite close together, however, and whereas Arndt [1] gives an SNR difference of 3 dB, our results imply that the minimum detectable signals do not differ by more than one dB.

MAXIMUM LIKELIHOOD BEARING ESTIMATES

If the filter output y of the multiplicative array model is to be operated upon to obtain an estimate of the bearing α, it is shown in [4] for uncorrelated noise the maximum likelihood bearing estimator has the approximate forms

$$\hat{\alpha}(y) = \tfrac{1}{2}\arccos\left[1 - \frac{48}{\beta^2(M^2-1)}\left(1 - \frac{s^2}{3Hy}\right)\right], \text{ low SNR} \qquad (8)$$

$$\hat{\alpha}(y) = \tfrac{1}{2}\arccos\left[1 - \frac{48}{\beta^2(4M^2-1)}\sqrt{\frac{1}{MH^2}\left(\frac{y}{Ms^2}+\frac{1}{2}\right)}\right], \text{ high SNR.} \qquad (9)$$

Also, the corresponding forms for the square-law model are

$$\hat{\alpha}(y) = \tfrac{1}{2}\arccos\left\{1 - \frac{48}{\beta^2(N^2-1)}\left[1 - \frac{s}{.52H\sqrt{y}}\ln\left(\frac{1.168y}{Ns^2}\right)\right]\right\}$$
$$\text{low SNR} \qquad (10)$$

$$\hat{\alpha}(y) = \arcsin\frac{2}{\beta}\sqrt{\frac{6}{N^2-1}\left(1 - \frac{\sqrt{y}}{NHs}\right)}, \text{ high SNR.} \qquad (11)$$

In [4] the mean values of these estimators for high SNR and for N = 2M are calculated to be

$$E(\hat{\alpha}) = 2kKe^{-h^2}\sum_{m=0}^{\infty}\frac{(-k)^m}{m!}B(\tfrac{1}{2},2m+2)L_m(h^2) \qquad (12)$$
$$\text{square law}$$

10

and $E(\hat{\alpha}) = K - KQ(\lambda, 2\sqrt{g}) + 2kK\exp(-\frac{1}{2}\lambda^2 - 2g)$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(g\lambda^2)^n(-k)^m}{n!(n+m)!} B(\tfrac{1}{2}, 2M+2) L_m^n(\tfrac{1}{2}\lambda^2), \qquad (13)$$

multiplicative

with $K = 4.9/\beta\sqrt{N^2-1}$, $k = NH^2$, $h^2 = NH^2D^2_{N/2}$. $\lambda^2 = 1 + 2h^2$, $4g = \lambda^2 - 2h^2\cos 2\theta$, and the B(,) are beta functions.

Computations of these mean values are given in Figures 11 and 12, with a straight line representing an unbiased estimator drawn in for reference. The trend, as the SNR increases, toward unbiasedness--where $E(\hat{\alpha}) = \alpha$,--is clearly seen for the smaller values of bearing. On the whole, the two estimators appear to behave quite similarly with respect to bias.

BOUNDS ON ESTIMATOR ERROR

As a means of learning estimator performance we have chosen the Cramer-Rao bounds. Using the above expressions for the mean values of the bearing estimates and working with the filter output probability density functions, we may calculate the familiar Cramer-Rao bounds on estimator variance:

$$Var(\hat{\alpha}) \geq [\frac{\partial}{\partial\alpha}E(\hat{\alpha})]^2/E\{[\frac{\partial \ln p(y;\alpha)}{\partial\alpha}]^2\}. \qquad (14)$$

An example of this bound for each array model is given in Figure 13, in which for small bearings the square-law array is seen to have a smaller ~~upper~~ lower bound on variance. It should be kept in mind that, since there is a bias, the mean square error will be greater than the variance. These curves, as well as others we have calculated for actual variance and mean square error, show a smaller error for the square-law array for small (major lobe) bearings, while the fact that the curves cross at higher values of bearing seems to imply that there are values at which the multiplicative array's estimator is superior.

CONCLUSION

We have very briefly summarized recent investigations which, by direct calculation, demonstrate that the actual relative performances of square-law and multiplicative array processors in signal detection and in bearing estimation can differ significantly from those predicted using the systems' SNR and beamwidth as figures of merit.

FIGURE 9. Receiver operating characteristics (N=4)

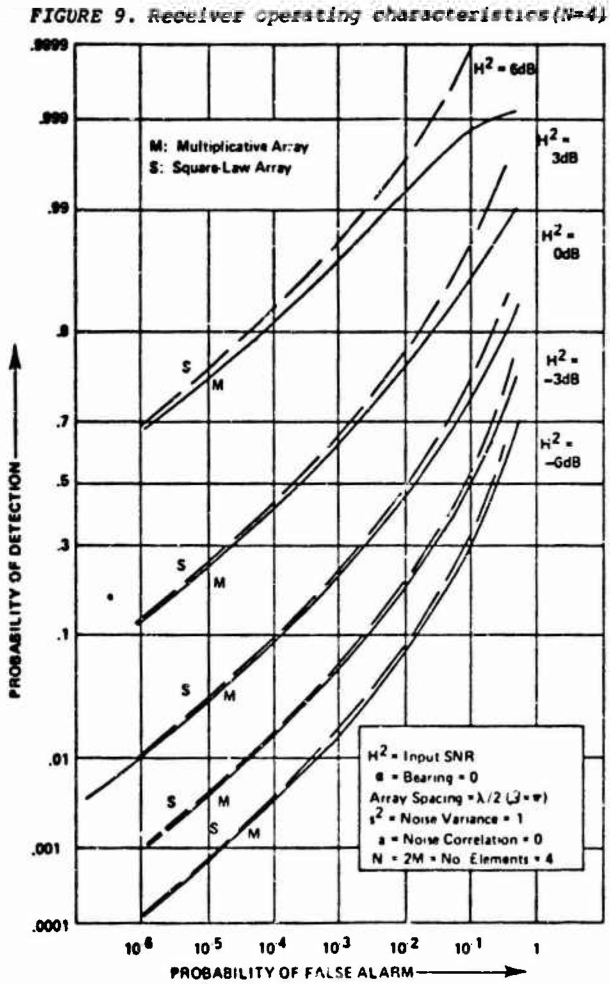
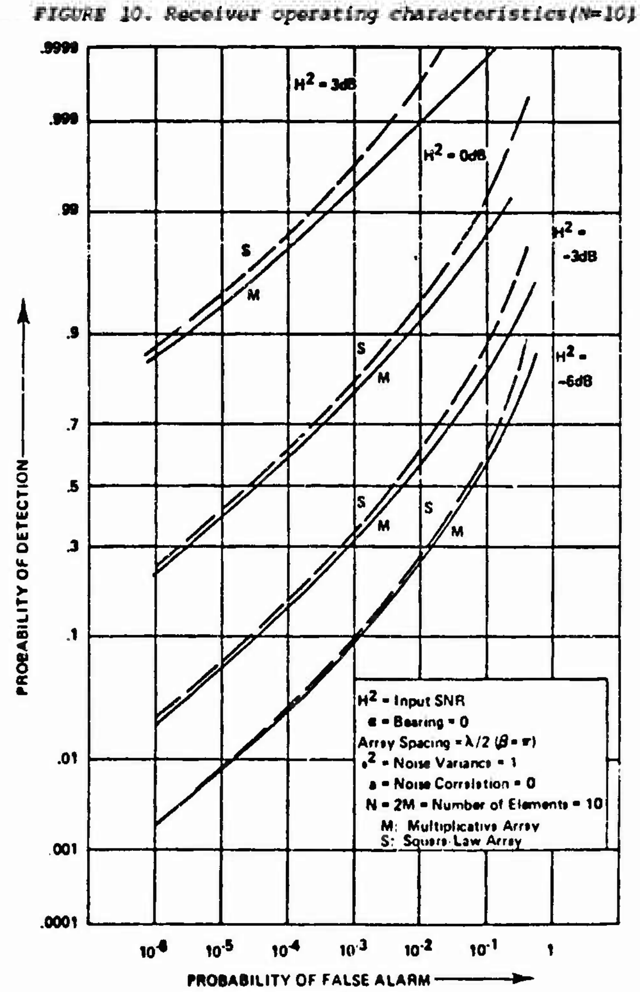


FIGURE 10. Receiver operating characteristics (N=10)



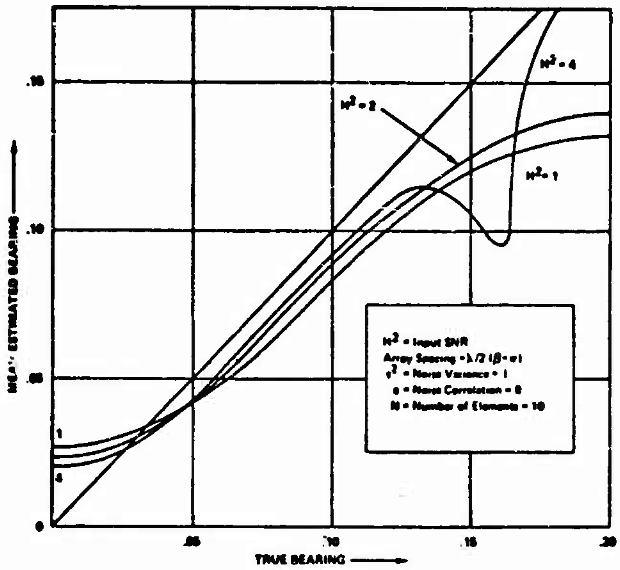


FIGURE 11. Mean value of square-law array's bearing estimate vs. actual bearing.

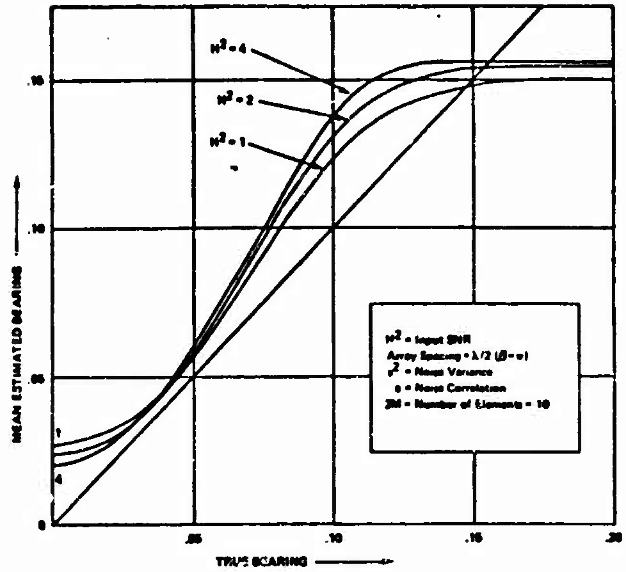


FIGURE 12. Mean value of multiplicative array's bearing estimate vs. actual bearing.

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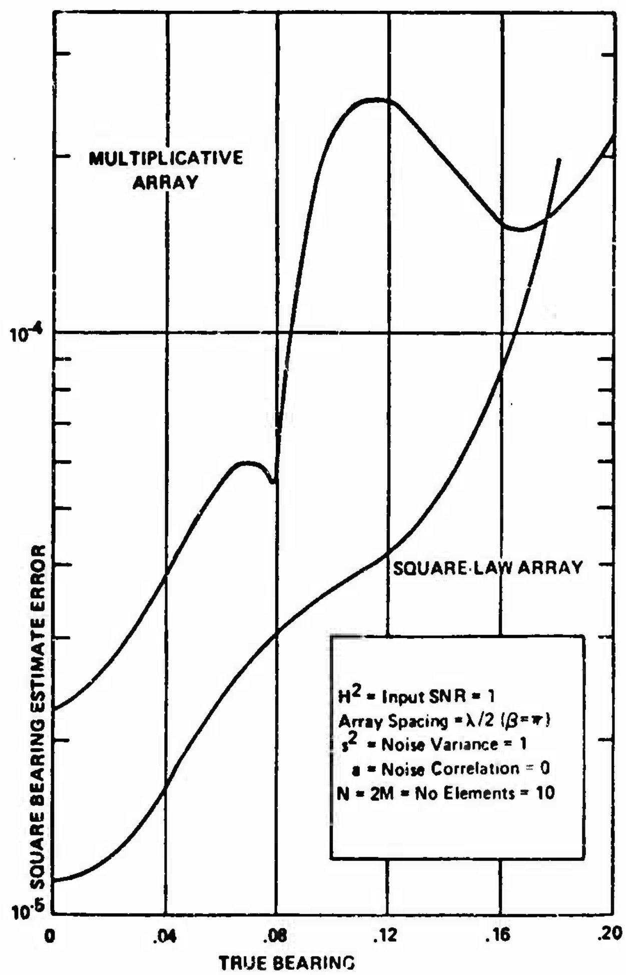
FIGURE 13. Cramer-Rao bounds on bearing estimation error vs. actual bearing.



THE PROBABILITY DENSITY FUNCTION FOR THE OUTPUT OF AN ANALOG CROSS-CORRELATOR WITH CORRELATED BANDPASS INPUTS

by

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The Probability Density Function for the Output of an Analog Cross-Correlator with Correlated Bandpass Inputs

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Abstract—The probability density function (pdf) for the output of an analog cross-correlator with correlated bandpass inputs is derived. The pdf is derived by a "direct method" without resorting to the "characteristic function method," which usually requires contour integrations in a complex plane for inversion operations. The correlator consists of bandpass filters, a multiplier, and a zonal low-pass filter. We treat the general situation in which the two inputs are narrow-band signals of unequal power and of different phases. The bandpass input noises are assumed to be correlated and may have different powers. In the Appendix, another derivation for the pdf is given in the special case of equal power correlated noise. This derivation is based on the fact that the correlator output random variable is the difference of two independent noncentral chi-square variables of two degrees of freedom. We show that the two expressions for the pdf (one from the direct method and the other from the characteristic function method) are indeed equivalent. Finally, we discuss two major areas of application.

I. Introduction

The PROBABILITY density function (pdf) for the output of an analog cross-correlator with correlated bandpass inputs is derived. The correlator consists of bandpass filters, a multiplier, and a zonal low-pass filter. We treat the general situation in which the two inputs are narrow-band signals of unequal power and of different phases. The noises in the input channels are assumed to be correlated with unequal power.

The problem of obtaining the pdf for the output of a cross-correlator with bandpass inputs has been a matter of continuing interest [1]–[7]. To reduce the complexity inherent in the mathematical aspects of the problem, previous authors have often restricted their considerations to certain simplifying assumptions or approximations in deriving a general expression for the pdf. However, Andrews [5] is an exception; he has obtained a general expression of considerable complexity by utilizing the characteristic function method. In this paper we show how the pdf can be derived in a direct fashion without employing the characteristic function method. The resultant expression for the pdf appears to be much simpler than that of Andrews. In the Appendix, yet another derivation of the pdf for the correlator output is given. This derivation is based on the fact that the correlator output random variable, manipulated suitably, is a difference of two independent noncentral chi-square variables, each with two degrees of freedom. Under this formulation for the output variable, the application of the characteristic function method gave an easy derivation for the pdf. These findings resulted from a study of the detection capabilities of narrowband multiplicative array configurations [8].

The "direct approach" we have used to obtain the filter output probability density function is by no means the only one. Perhaps the most thorough method is that which takes into account the filter transfer function of a realizable filter and treats the filter output as a time series. In the literature, this approach is often attributed to Kac and Siegert [9], and usually is found applied to the square-law detector [10]–[12]. An exception is the work of Lampard [3], who studies a multiplier/filter combination.

Other references which use the same "zonal filter" model as we do to analyze the multiplier/filter, though less generally, are [1], [2], [4], and [5]. A thorough treatment of the output SNR of such systems is that of Green [13]. In this area, very similar pdf curves were obtained from sometimes very dissimilar expressions.

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II. The Cross-Correlator and Formulation of the Problem

The cross-correlator to be studied is shown in Fig. 1. The inputs to the multiplier are two narrow-band processes $u_1(t)$ and $u_2(t)$, and their product $z(t)$ is passed through a zonal lowpass filter to yield the output $y(t)$, as shown. The processes $u_1(t)$ and $u_2(t)$ are said to consist of the superposition of the deterministic signals $s_1(t)$, $s_2(t)$ and the noise processes $n_1(t)$, $n_2(t)$, respectively. At the same instant of time, $n_1(t)$ and $n_2(t)$ are assumed to be jointly Gaussian random variables with variances σ_1^2 and σ_2^2, correlation coefficient ρ, and zero means.

The multiplier is assumed to be instantaneous, so that its output can be written

$$\begin{aligned} z(t) &= u_1(t) \times u_2(t) \\ &= [s_1(t) + n_1(t)] \times [s_2(t) + n_2(t)] \\ &= \tfrac{1}{4}[u_1(t) + u_2(t)]^2 - \tfrac{1}{4}[u_1(t) - u_2(t)]^2 \\ &\equiv [s_3(t) + n_3(t)]^2 - [s_4(t) + n_4(t)]^2 \qquad (1) \end{aligned}$$

where

$$s_{3,4}(t) = \tfrac{1}{2}[s_1(t) \pm s_2(t)] \qquad (1a)$$

and

$$n_{3,4}(t) = \tfrac{1}{2}[n_1(t) \pm n_2(t)]. \qquad (1b)$$

This arrangement may be recognized as the old "quarter-square multiplier" idea used in analog computation.

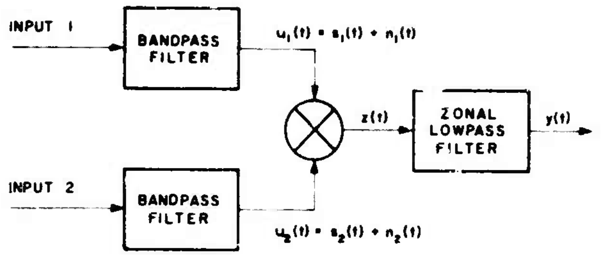

Fig. 1. Block diagram of cross-correlator.

For the new variables we have defined, we have the following moments:

$$E(n_3) = 0$$
$$E(n_4) = 0 \tag{2a}$$
$$E(n_3^2) \triangleq \sigma_3^2 = \tfrac{1}{4}(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)$$
$$E(n_4^2) \triangleq \sigma_4^2 = \tfrac{1}{4}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)$$
$$E(n_3 n_4) \triangleq R\sigma_3\sigma_4 = \tfrac{1}{4}(\sigma_1^2 - \sigma_2^2). \tag{2b}$$

Note that if $\sigma_1 = \sigma_2$, then $R = 0$; that is, n_3 and n_4 are uncorrelated.

Specialization to what we are calling the narrow-band case consists in the conventional assumption that the following quadrature representations apply:

$$\begin{aligned} s_3(t) &= S_3(t)\cos[\omega t - \theta_3(t)] \\ s_4(t) &= S_4(t)\cos[\omega t - \theta_4(t)] \\ n_3(t) &= n_{3c}(t)\cos\omega t + n_{3s}(t)\sin\omega t \\ &= N_3(t)\cos[\omega t - \phi_3(t)] \\ n_4(t) &= n_{4c}(t)\cos\omega t + n_{4s}(t)\sin\omega t \\ &= N_4(t)\cos[\omega t - \phi_4(t)] \end{aligned} \tag{3}$$

where if

$$\begin{aligned} s_i(t) &= S_i(t)\cos[\omega t - \theta_i(t)] \\ n_i(t) &= N_i(t)\cos[\omega t - \phi_i(t)] \end{aligned} \tag{3a}$$

where $i = 1,2$ then

$$S_{3,4}^2 = \tfrac{1}{4}[S_1^2 + S_2^2 \pm 2S_1S_2\cos(\theta_1 - \theta_2)]$$
$$\tan\theta_{3,4} = \frac{S_1\sin\theta_1 \pm S_2\sin\theta_2}{S_1\cos\theta_1 \pm S_2\cos\theta_2}$$
$$N_{3,4}^2 = \tfrac{1}{4}[N_1^2 + N_2^2 \pm 2N_1N_2\cos(\phi_1 - \phi_2)]$$
$$\tan\phi_{3,4} = \frac{N_1\sin\phi_1 \pm N_2\sin\phi_2}{N_1\cos\phi_1 \pm N_2\cos\phi_2}. \tag{3b}$$

With this narrow-band representation, we have for the output of the multiplier

$$\begin{aligned} z &= (s_3 + n_3)^2 - (s_4 + n_4)^2 \\ &= \tfrac{1}{2}[S_3^2 + 2S_3N_3\cos(\phi_3 - \theta_3) + N_3^2 - S_4^2 \\ &\quad - 2S_4N_4\cos(\phi_4 - \theta_4) - N_4^2] \\ &\quad + (\text{terms with frequency } 2\omega). \end{aligned} \tag{4}$$

What is meant by "zonal low-pass filter" is that the filter output y is equal to the multiplier output z, less the terms of frequency 2ω. Thus if we define the sum and difference terms

$$\begin{aligned} z_1(t) &\triangleq s_3(t) + n_3(t) = \tfrac{1}{2}[u_1(t) + u_2(t)] \\ &= Z_1(t)\cos[\omega t - \phi_1(t)] \\ &= z_{1c}(t)\cos\omega t + z_{1s}(t)\sin\omega t \end{aligned} \tag{4a}$$

$$\begin{aligned} z_2(t) &\triangleq s_4(t) + n_4(t) = \tfrac{1}{2}[u_1(t) - u_2(t)] \\ &= Z_2(t)\cos[\omega t - \phi_2(t)] \\ &= z_{2c}(t)\cos\omega t + z_{2s}(t)\sin\omega t \end{aligned} \tag{4b}$$

we find that

$$y(t) = \tfrac{1}{2}[Z_1^2(t) - Z_2^2(t)] \tag{5}$$

where

$$\begin{aligned} Z_{1,2}^2 &= \tfrac{1}{4}[S_1^2 + S_2^2 + N_1^2 + N_2^2 \pm 2S_1S_2\cos(\theta_2 - \theta_1) \\ &\quad + 2S_1N_1\cos(\phi_1 - \theta_1) \pm 2S_1N_2\cos(\phi_2 - \theta_1) \\ &\quad \pm 2S_2N_1\cos(\phi_1 - \theta_2) + 2S_2N_2\cos(\phi_2 - \theta_2) \\ &\quad \pm 2N_1N_2\cos(\phi_1 - \phi_2)] \\ &= S_{3,4}^2 + N_{3,4}^2 + 2S_{3,4}N_{3,4}\cos(\phi_{3,4} - \theta_{3,4}) \end{aligned} \tag{5a}$$

and

$$\begin{aligned} \tan\phi_{1,2} &= \frac{S_1\sin\theta_1 \pm S_2\sin\theta_2 + N_1\sin\phi_1 \pm N_2\sin\phi_2}{S_1\cos\theta_1 \pm S_2\cos\theta_2 + N_1\cos\phi_1 \pm N_2\cos\phi_2} \\ &= \frac{S_{3,4}\sin\theta_{3,4} + N_{3,4}\sin\phi_{3,4}}{S_{3,4}\cos\theta_{3,4} + N_{3,4}\cos\phi_{3,4}}. \end{aligned} \tag{5b}$$

Our object is to find the pdf of the filter output $y(t)$ at a given instant in time. (From this point on, reference to time will be suppressed.)

Distribution of the Sum and Difference Terms

If, to use vector notation, we refer to the input noise components as $x' = (n_{1c}, n_{1s}, n_{2c}, n_{2s})$, where the prime (′) is used to indicate the transpose, then we may express the joint pdf of the input noise components as

$$p_0(x) = \frac{1}{4\pi^2\sqrt{\det K_x}}\exp[-\tfrac{1}{2}x'K_x^{-1}x]$$

where K_x, the covariance matrix, is postulated to be

$$\begin{aligned} K_x &= \operatorname{cov}[n_{1c}(t), n_{1s}(t), n_{2c}(t), n_{2s}(t)] \\ &= E[x'x] \\ &\equiv \begin{bmatrix} \sigma_1^2 & 0 & \rho\sigma_1\sigma_2 & r\sigma_1\sigma_2 \\ 0 & \sigma_1^2 & -r\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & -r\sigma_1\sigma_2 & \sigma_2^2 & 0 \\ r\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 & 0 & \sigma_2^2 \end{bmatrix}. \end{aligned}$$

That is, the noise correlation at the inputs to the multiplier is considered to be such that, for the same time instant

$$E(n_{1c}n_{2c}) = E(n_{1s}n_{2s}) = \rho\sigma_1\sigma_2$$
$$E(n_{1c}n_{2s}) = -E(n_{1s}n_{2c}) = r\sigma_1\sigma_2.$$

We may consider the sum and difference noise components $n' = (n_{3c}, n_{3s}, n_{4c}, n_{4s})$ as a linear transformation $n = Ax$, where the matrix A is given by

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

We may then write the pdf of the variables n as

$$p_1(n) = \frac{1}{4\pi^2 D} \exp\left[-\tfrac{1}{2} n' K_n^{-1} n\right] \tag{6}$$

where the new covariance matrix K_n is easily computed to be

$$K_n = AK_xA$$
$$= \begin{bmatrix} \sigma_3^2 & 0 & R\sigma_3\sigma_4 & R'\sigma_3\sigma_4 \\ 0 & \sigma_3^2 & -R'\sigma_3\sigma_4 & R\sigma_3\sigma_4 \\ R\sigma_3\sigma_4 & -R'\sigma_3\sigma_4 & \sigma_4^2 & 0 \\ -R'\sigma_3\sigma_4 & R\sigma_3\sigma_4 & 0 & \sigma_4^2 \end{bmatrix} \tag{7}$$

$$D = \sqrt{\det K_n} = \sigma_3^2\sigma_4^2[1 - R^2 - (R')^2]. \tag{7a}$$

The variances σ_3^2 and σ_4^2, and R are as given in (2), and $R'\sigma_3\sigma_4 = -\tfrac{1}{2} r\sigma_1\sigma_2$.

It follows then that the quadrature expansions of the signal plus noise variables z_1 and z_2 as just defined have the density function

$$p_2(z_{1c}, z_{1s}, z_{2c}, z_{2s}) = \frac{1}{4\pi^2 D} \exp\left[-\tfrac{1}{2}(z - s)' K_n^{-1}(z - s)\right] \tag{8}$$

where

$$(z - s)' = (z_{1c} - S_3 \cos\theta_3, z_{1s} - S_3 \sin\theta_3,$$
$$z_{2c} - S_4 \cos\theta_4, z_{2s} - S_4 \sin\theta_4). \tag{9}$$

III. General Expression of the Probability Density Function for the Correlator Output

Recall that the filter output is $y = \tfrac{1}{2}(Z_1^2 - Z_2^2)$. The density function of Z_1 and Z_2 is

$$p_4(Z_1, Z_2) = \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2\, p_3(Z_1, \phi_1, Z_2, \phi_2)$$
$$= Z_1 Z_2 \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2\, p_2(Z_1 \cos\phi_1, Z_1 \sin\phi_1,$$
$$Z_2 \cos\phi_2, Z_2 \sin\phi_2) \tag{10a}$$
$$= \frac{Z_1 Z_2}{4\pi^2 D} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2$$
$$\cdot \exp\left[-\tfrac{1}{2}(Z - s)' K_n^{-1}(Z - s)\right] \tag{10b}$$

where

$$(Z - s)'$$
$$= (Z_1 \cos\phi_1 - S_3 \cos\theta_3, Z_1 \sin\phi_1 - S_3 \sin\theta_3,$$
$$Z_2 \cos\phi_2 - S_4 \cos\theta_4, Z_2 \sin\phi_2 - S_4 \sin\theta_4). \tag{11}$$

The quadratic form appearing in the integral's exponent reduces to

$$Q = \frac{-1}{2D}\,[\sigma_4^2(Z_1^2 + S_3^2) + \sigma_3^2(Z_2^2 + S_4^2)$$
$$- 2\sigma_3\sigma_4 X S_3 S_4 \cos(\theta_4 - \theta_3 - x)$$
$$- 2\sigma_3\sigma_4 X Z_1 Z_2 \cos(\phi_1 - \phi_2 - x)$$
$$- 2\sigma_4 V Z_1 \cos(\phi_1 - v) - 2\sigma_3 W Z_2 \cos(\phi_2 - w)] \tag{12}$$

where

$$X^2 = R^2 + (R')^2 \qquad \tan x = R'/R$$
$$V^2 = \sigma_4^2 S_3^2 + \sigma_3^2 X^2 S_4^2$$
$$- 2\sigma_3\sigma_4 X S_3 S_4 \cos(\theta_3 - \theta_4 + x)$$
$$W^2 = \sigma_3^2 S_4^2 + \sigma_4^2 X^2 S_3^2$$
$$- 2\sigma_3\sigma_4 X S_3 S_4 \cos(\theta_3 - \theta_4 + x)$$
$$\tan v = \frac{\sigma_4 S_3 \sin\theta_3 - \sigma_3 X S_4 \sin(\theta_4 - x)}{\sigma_4 S_3 \cos\theta_3 - \sigma_3 X S_4 \cos(\theta_4 - x)}$$
$$\tan w = \frac{\sigma_3 S_4 \sin\theta_4 - \sigma_4 X S_3 \sin(\theta_3 + x)}{\sigma_3 S_4 \cos\theta_4 - \sigma_4 X S_3 \cos(\theta_3 + x)}. \tag{13}$$

Using the relationship

$$e^{x\cos w} = \sum_{n=0}^{\infty} \varepsilon_n I_n(x) \cos nw, \qquad \varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0 \end{cases}$$

and performing the integration with respect to ϕ_1 and ϕ_2, in a manner quite similar to Middleton [14, ch. 9] we obtain

$$p_4(Z_1, Z_2) = \frac{Z_1 Z_2}{D} \exp\left\{\frac{-1}{2D}\,[\sigma_4^2 Z_1^2 + \sigma_3^2 Z_2^2 + U^2]\right\}$$
$$\cdot \sum_{m=0}^{\infty} \varepsilon_m I_m\left(\frac{\sigma_4 V Z_1}{D}\right) I_m\left(\frac{\sigma_3 W Z_2}{D}\right)$$
$$\cdot I_m\left(\frac{\sigma_3\sigma_4 X Z_1 Z_2}{D}\right) \cos m(v - w - x) \tag{14}$$

where we have used

$$U^2 = \sigma_4^2 S_3^2 + \sigma_3^2 S_4^2 - 2\sigma_3\sigma_4 X S_3 S_4 \cos(\theta_4 - \theta_3 - x), \tag{15}$$

and $I_m(\cdot)$ is the mth order modified Bessel function of the first kind.

To obtain the density for y, we use the transformations

$$Z_1 = \sqrt{2y}\cosh u,$$
$$Z_2 = \sqrt{2y}\sinh u, \qquad \text{for } Z_1 > Z_2, \text{ or } y > 0 \text{ and } u > 0$$

and

$$Z_1 = \sqrt{-2y}\sinh u,$$
$$Z_2 = \sqrt{-2y}\cosh u, \qquad \text{for } Z_1 < Z_2, \text{ or } y < 0 \text{ and } u > 0. \tag{16}$$

Then we find that the density for y is given by the expression

$$p_6(y) = \int_0^\infty du\, p_5(y,u)$$

$$= \begin{cases} \int_0^\infty du\, p_4(\sqrt{2y}\cosh u, \sqrt{2y}\sinh u), & y > 0 \\ \int_0^\infty du\, p_4(\sqrt{-2y}\sinh u, \sqrt{-2y}\cosh u), & y < 0 \end{cases}$$

$$= \frac{|y|}{D}\exp\left[\frac{-(\sigma_3^2 - \sigma_4^2)|y| - U^2}{2D}\right]$$

$$\cdot \sum_{m=0}^{\infty} \varepsilon_m \cos m(v - w - x) \int_0^\infty du \sinh 2u\, e^{-a\cosh 2u}$$

$$\cdot I_m(b\cosh u) I_m(c\sinh u) I_m(d\sinh 2u) \tag{17}$$

with

$$a = |y|(\sigma_3^2 + \sigma_4^2)/2D \qquad d = \sigma_3\sigma_4 X|y|/D$$

and

$$(b,c) = (\sigma_4 V\sqrt{2y}/D,\ \sigma_3 W\sqrt{2y}/D), \qquad y > 0$$

$$= (\sigma_3 W\sqrt{-2y}/D,\ \sigma_4 V\sqrt{-2y}/D), \qquad y < 0. \tag{18}$$

By writing $x = 1 + \cosh 2u$, we may change the integral to

$$\frac{1}{2}e^a \int_0^\infty dx\, e^{-ax} I_m[b\sqrt{\tfrac{1}{2}(x+2)}] I_m[c\sqrt{\tfrac{1}{2}x}] I_m[d\sqrt{x(x+2)}]$$

$$= \frac{1}{2}e^a \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(b/2\sqrt{2})^{2n+m}}{n!\,(n+m)!}\frac{(\frac{1}{2}d)^{2k+m}}{k!\,(k+m)!}$$

$$\cdot \int_0^\infty dx\, e^{-ax}(x+2)^{n+k+m} x^{k+(1/2)m} I_m[c\sqrt{\tfrac{1}{2}x}]$$

$$= \frac{1}{2}e^a \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n+k+m} \frac{(b/2\sqrt{2})^{2n+m}}{n!\,(n+m)!}\frac{(\frac{1}{2}d)^{2k+m}}{k!\,(k+m)!}$$

$$\cdot \binom{n+k+m}{r} 2^{n+k+m-r}$$

$$\cdot \int_0^\infty dx\, e^{-ax} x^{k+(1/2)m+r} I_m[c\sqrt{\tfrac{1}{2}x}]. \tag{19}$$

Making use of Gradshteyn and Ryzhik [15, formula 6.643.4], we obtain the following expression for (19):

$$\left(\frac{c}{2a}\right)^m \frac{1}{2a}\exp\left(a + \frac{c^2}{8a}\right)$$

$$\cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n+k+m} \frac{(\frac{1}{2}b)^{2n+m}(\frac{1}{2}d)^{2k+m}}{n!\,(n+m)!\,k!\,(k+m)!}$$

$$\cdot \binom{n+k+m}{r}\frac{2^{k-r}}{a^{k+r}}(k+r)!\, L_{k+r}^m\left(\frac{-c^2}{8a}\right) \tag{20}$$

where the $L_{k+r}^m(\cdot)$ are the Laguerre polynomials of order $k + r$ and parameter m. Substituting in (17) and for a, b, c, and d, and rearranging terms for clarity, we have finally

$$p_6(y) = \frac{1}{\sigma_3^2 + \sigma_4^2}\exp\left\{\frac{-1}{2D}\left[2\sigma_0^2|y| + U^2 - \frac{g^2}{\sigma_3^2 + \sigma_4^2}\right]\right\}$$

$$\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n+k+m}\left(\frac{fg\sigma_3\sigma_4 X}{2D^2(\sigma_3^2 + \sigma_4^2)}\right)^m$$

$$\cdot \left(\frac{f^2}{2D^2}\right)^n \left(\frac{\sigma_3^2\sigma_4^2 X^2}{D(\sigma_3^2 + \sigma_4^2)}\right)^k \left(\frac{D}{\sigma_3^2 + \sigma_4^2}\right)^r$$

$$\cdot \varepsilon_m \cos m(v - w - x)\frac{(k+r)!}{n!\,(n+m)!\,k!\,(k+m)!}$$

$$\cdot \binom{k+n+m}{r} |y|^{k+m+n-r} L_{k+r}^m\left(\frac{-g^2}{2D(\sigma_3^2 + \sigma_4^2)}\right) \tag{21}$$

with

$$(f,g,\sigma_0^2) = \begin{cases}(\sigma_4 V, \sigma_3 W, \sigma_4^2), & y > 0 \\ (\sigma_3 W, \sigma_4 V, \sigma_3^2), & y < 0.\end{cases} \tag{22}$$

A recent paper by Andrews [5] gives an expression for the density function in nearly the same situation by means of the characteristic function method. Comparing the development here with that of Andrews, it appears that following the strenuous development associated with the computations of the characteristic function, one still needs to solve a convolution integral such as our (19). The different final expression he obtains results simply from a different method of attacking this type of integral. For the special cases, of course, the form of the expressions given here is identical to that of Andrews. Moreover, the computed results we show, match his wherever applicable.

IV. Reduction of the General Expression to Special Cases

For practical applications and for purposes of checking our results with those of previous authors, it is instructive to reduce the general expression of (21) under certain assumptions.

A. Equal Noise Power with Even Spectrum

An important case is that for which the noise inputs are of equal power and their spectra are even about the center frequency of the band. Under these assumptions, the cross-quadrature correlation coefficient r is zero [14] and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, implying that $R = R' = 0$, so that the term X in (21) is zero. Defining the input SNR for channels 1 and 2 by

$$h_1^2 \triangleq \frac{S_1^2}{2\sigma^2}$$

and

$$h_2^2 \triangleq \frac{S_2^2}{2\sigma^2}$$

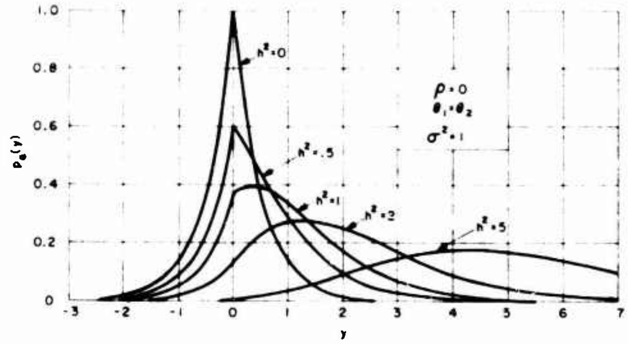

Fig. 2. Probability density functions for cross-correlator output $y(t)$ for several values of identical input SNR's when $\rho = 0$, $\theta_1 = \theta_2$, and $\sigma^2 = 1$.

the general expression of (21) is reduced to the following form for $y > 0$:

$$p_6(y) = \frac{1}{\sigma^2} \exp\left\{-\frac{2y}{\sigma^2(1+\rho)} - h_3^2 - \left(\frac{1-\rho}{2}\right) h_4^2\right\}$$
$$\cdot \sum_{r=0}^{\infty} \left[\frac{h_3(1-\rho)\sqrt{1+\rho}}{\sqrt{8y/\sigma^2}}\right]^r I_r\left(\frac{2h_3}{\sigma}\sqrt{\frac{2y}{1+\rho}}\right)$$
$$\cdot L_r\left[-\left(\frac{1+\rho}{2}\right) h_4^2\right] \tag{23a}$$

and for $y < 0$

$$p_6(y) = \frac{1}{\sigma^2} \exp\left\{\frac{2y}{\sigma^2(1-\rho)} - \left(\frac{1+\rho}{2}\right) h_3^2 - h_4^2\right\}$$
$$\cdot \sum_{r=0}^{\infty} \left[\frac{h_4(1+\rho)\sqrt{1-\rho}}{\sqrt{-8y/\sigma^2}}\right]^r I_r\left(\frac{2h_4}{\sigma}\sqrt{\frac{-2y}{1-\rho}}\right)$$
$$\cdot L_r\left[-\frac{(1-\rho)}{2} h_3^2\right] \tag{23b}$$

where

$$h_3^2 = \frac{1}{2(1+\rho)} (h_1^2 + h_2^2 + 2h_1h_2 \cos 2\theta) \tag{23c}$$

$$h_4^2 = \frac{1}{2(1-\rho)} (h_1^2 + h_2^2 - 2h_1h_2 \cos 2\theta) \tag{23d}$$

and where $2\theta = \theta_1 - \theta_2$ is the difference in phase between the narrow-band signals at the multiplier input (see Fig. 1 and (3a)). Muraka [6] presents results corresponding to a further specialization of this case to $\theta = 0$ and $h_1^2 = h_2^2$.

B. No Input Signals and Equal Noise Power

If the correlator inputs are assumed to be only the noise of equal power, that is, $h_1^2 = h_2^2 = 0$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the pdf (21) is further reduced to

$$p_6(y) = \begin{cases} \dfrac{1}{\sigma^2} \exp\left[-\dfrac{2y}{\sigma^2(1+\rho)}\right], & y > 0 \\[2ex] \dfrac{1}{\sigma^2} \exp\left[\dfrac{2y}{\sigma^2(1-\rho)}\right], & y < 0. \end{cases} \tag{24}$$

This result is identical to that of Lezin [7], who began analysis with these assumptions.

C. Computed Results for Special Cases

Judging the equal power ($\sigma_1^2 = \sigma_2^2 = \sigma^2$), even spectrum case to be of sufficient interest, we have plotted several curves for $h_1 = h_2 = h$ in order to show the effects of the parameters ρ, h^2, and $\theta = (\theta_1 - \theta_2)/2$; σ, quite obviously, is a scaling parameter. In the Appendix the pdf (23) is shown to be equivalent to

$$p_6(y) = \frac{1}{\sigma^2} \exp\left[-(h_3^2 + h_4^2)\right]$$
$$\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{[\frac{1}{2}(1-\rho)h_3^2]^m}{m!} \frac{[\frac{1}{2}(1+\rho)h_4^2]^n}{n!}$$
$$\cdot \begin{cases} \exp\left[\dfrac{-2y}{\sigma^2(1+\rho)}\right] G_m^n\left[\dfrac{4y}{\sigma^2(1-\rho^2)}\right], & y \geq 0 \\[2ex] \exp\left[\dfrac{2y}{\sigma^2(1-\rho)}\right] G_n^m\left[\dfrac{-4y}{\sigma^2(1-\rho^2)}\right], & y < 0 \end{cases} \tag{25}$$

where the polynomials we have defined,[1]

$$G_m^n(x) = \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k}{k!}$$

have the unusually useful computational property $G_m^n = G_{m-1}^n + G_m^{n-1}$. The form (25) results from application of the characteristic function method to the special case we are now considering.

For computation of (25), we have chosen a "nominal case" of specific values for the parameters: $\rho = 0$, $\sigma = 1$, $h = 1$, $\theta = 0$. In Fig. 2, the parameter h^2 (SNR going into the multiplier) is varied. It is seen to have the powerful effect of changing the pointed curve of the no-signal case into curves which begin to approach the familiar Gaussian.

The correlation coefficient at the multiplier inputs ρ is varied in Fig. 3, where it is evident that increasing correlation produces a more "noiselike" distribution, that is, one with a smaller apparent SNR. Figs. 2 and 3 include curves which exactly match those given by Andrews [5], although the scaling is somewhat different due to his inclusion of a scale factor in the definition of the multiplier.

The phase between the two multiplier signal inputs θ is varied in Fig. 4. The effect of this parameter is seen to shift the concentration of probable values closer to zero, decreasing the apparent SNR. Note that the cases $\theta = \pi/6$ and $\theta = \pi/3$ are "reflections" of one another.

V. Remarks on the Applications of the Results

The pdf for the output of a cross-correlator with bandpass inputs arises in many situations of practical interest. The equivalent pdf's (23) and (25) have been applied to the

[1] A further discussion on the properties of this polynomial $G_m^n(x)$ is given in the Appendix.

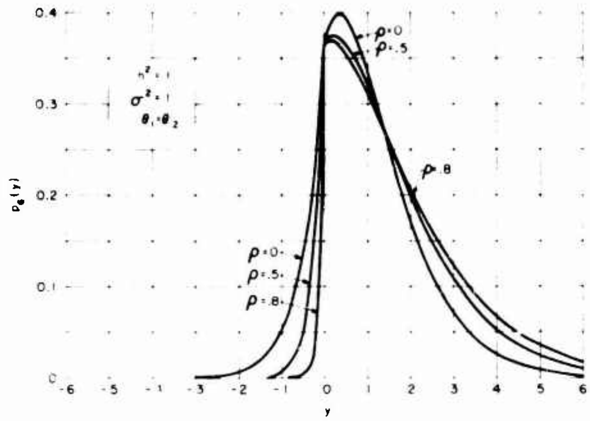

Fig. 3. Probability density functions for cross-correlator output $y(t)$ for several values of noise correlation coefficients when $h^2 = 1$, $\sigma^2 = 1$, and $\theta_1 = \theta_2$.

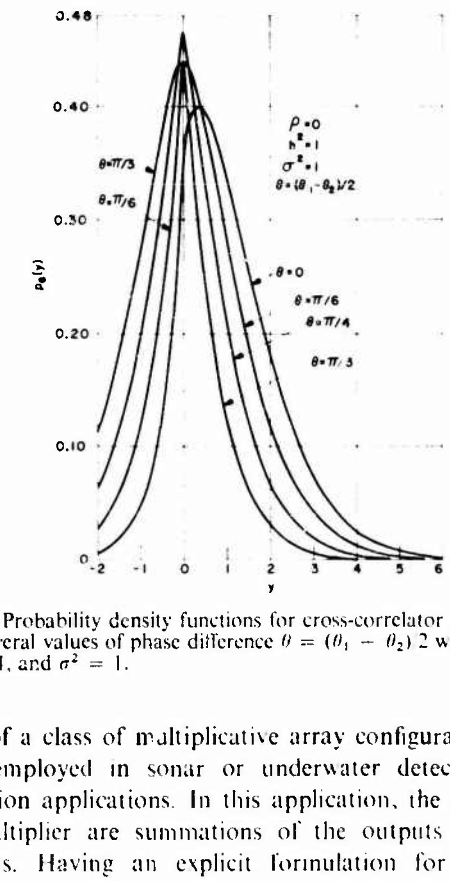

Fig. 4. Probability density functions for cross-correlator output $y(t)$ for several values of phase difference $\theta = (\theta_1 - \theta_2)/2$ when $\rho = 0$, $h^2 = 1$, and $\sigma^2 = 1$.

study of a class of multiplicative array configurations [8] often employed in sonar or underwater detection and estimation applications. In this application, the inputs to the multiplier are summations of the outputs of array elements. Having an explicit formulation for the pdf permitted direct calculation of the effects on the distribution of parameters such as signal bearing, the number of array elements, interelement correlation, array spacing, and input SNR.

Calculation of receiver operating characteristics (ROC) based on our results revealed that the multiplier/filter combination has the potential for performing signal detection very nearly as well as the more conventional square-law array configuration, whereas previous analysis based on SNR alone as a figure of merit predicted a 3 dB superiority for the conventional system. This particular application has been summarized in [22].

For those who wish to pursue the mathematical aspect of this type of problem, consult [23]–[30].

Application of the pdf for the correlator output in problems of digital communication is also rewarding. The correlator model we have considered corresponds to the receiver structure for binary communication in which the detection is accomplished with a noisy reference. The performance (error probability) computation, when the noises between the input and the reference channels are correlated, can be computed using the pdf of (23) for $h_1^2 = h_2^2 = h^2$. Cooper [2] has shown that the application of the pdf for the correlator output to binary orthogonal communication systems produces the same known result of error probability under uncorrelated noise ($\rho = 0$) assumptions.

Acknowledgment

The authors wish to express thanks to the referees, who suggested the relationship of the polynomials $G_m^n(x)$ to other special functions and who drew attention to [28]–[30].

Appendix

Another Form for the Probability Density Function

The case we are considering is that in which the cross-correlation coefficient $r = 0$ and $\sigma_1 = \sigma_2$ ($R \to 0$), so that the term X in (21) equals zero. Under these conditions, the sum and difference envelopes Z_1 and Z_2 are statistically independent, since we may factor their joint probability density function (14)

$$p_4(Z_1,Z_2) = \frac{Z_1}{\sigma_3^2} \exp\left(-\frac{Z_1^2 + S_3^2}{2\sigma_3^2}\right) I_0\left(\frac{S_3 Z_1}{\sigma_3^2}\right) \frac{Z_2}{\sigma_4^2} \cdot \exp\left(-\frac{Z_2^2 + S_4^2}{2\sigma_4^2}\right) I_0\left(\frac{S_4 Z_2}{\sigma_4^2}\right)$$

$$= \frac{2Z_1}{\sigma_3^2} \chi^2\left[\frac{Z_1^2}{\sigma_3^2}; 2, 2h_3^2\right] \frac{2Z_2}{\sigma_4^2} \chi^2\left[\frac{Z_2^2}{\sigma_4^2}; 2, 2h_4^2\right] \tag{26}$$

where $\sigma_3^2 = \frac{1}{2}\sigma^2(1 + \rho)$, $\sigma_4^2 = \frac{1}{2}\sigma^2(1 - \rho)$, $h_3^2 = S_3^2/2\sigma_3^2 = (h_1^2 + h_2^2 + 2h_1h_2 \cos 2\theta)/2(1 + \rho)$, and $h_4^2 = S_4^2/2\sigma_4^2 = (h_1^2 + h_2^2 - 2h_1h_2 \cos 2\theta)/2(1 - \rho)$. $\chi^2(x;2,a^2)$ is the noncentral chi-square distribution with two degrees of freedom and noncentrality parameter a^2. Thus the filter output $y = \frac{1}{2}(Z_1^2 - Z_2^2)$ is the difference between two scaled, independent noncentral chi-square random variables. This type of relationship was noted also by Lee [16] in a similar application.

The noncentral chi-square distribution has many forms and interpretations (see [17]–[20]). For example, we may write the following

$$\chi^2(x;2,a^2) = \exp(-\tfrac{1}{2}a^2) \sum_{r=0}^{\infty} \frac{(\frac{1}{2}a^2)^r}{r!} \chi^2(x; 2 + 2r) \tag{27}$$

where $\chi^2(x;n)$ is the (central) chi-square distribution with n degrees of freedom

$$\chi^2(x;n) = \frac{1}{2} e^{-x/2} \left(\frac{x}{2}\right)^{(n-2)/2} \left[\Gamma\left(\frac{n}{2}\right)\right]^{-1}. \tag{28}$$

A. Density for y Using Characteristic Function

Using an expression given by Jayachandran and Barr [21], we may write the characteristic function of the distribution of y as

$$\phi_y(t) = \exp\left[-(h_3^2 + h_4^2)\right] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(h_3^2)^m}{m!} \frac{(h_4^2)^n}{n!} \cdot \left\{ \left(\frac{1+\rho}{2}\right)^{n+1} \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{[\frac{1}{2}(1-\rho)]^{m-k}}{(1 - i\sigma_3^2 t)^{k+1}} + \left(\frac{1-\rho}{2}\right)^{m+1} \sum_{r=0}^{n} \binom{m+n-r}{m} \frac{[\frac{1}{2}(1+\rho)]^{n-r}}{(1 + i\sigma_4^2 t)^{r+1}} \right\}. \tag{29}$$

Thus we obtain for the density of y

$$p_6(y) = \frac{2}{\sigma^2} \exp\left[-(h_3^2 + h_4^2)\right] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[\frac{1}{2}(1-\rho)h_3^2]^m}{m!} \cdot \frac{[\frac{1}{2}(1+\rho)h_4^2]^n}{n!} \cdot \begin{cases} \displaystyle\sum_{k=0}^{m} \binom{m+n-k}{n} \left(\frac{1-\rho}{2}\right)^{-k} \chi^2\left[\frac{4y}{\sigma^2(1+\rho)};\ 2+2k\right], & y > 0 \\ \displaystyle\sum_{r=0}^{n} \binom{m+n-r}{m} \left(\frac{1+\rho}{2}\right)^{-r} \chi^2\left[\frac{-4y}{\sigma^2(1-\rho)};\ 2+2r\right], & y < 0. \end{cases} \tag{30}$$

Defining the polynomials

$$G_m^n(x) = \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k}{k!}$$

we finally obtain

$$p_6(y) = \frac{1}{\sigma^2} \exp\left[-(h_3^2 + h_4^2)\right] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{[\frac{1}{2}(1-\rho)h_3^2]^m}{m!} \cdot \frac{[\frac{1}{2}(1+\rho)h_4^2]^n}{n!} \cdot \begin{cases} \exp\left[\dfrac{-2y}{\sigma^2(1+\rho)}\right] G_m^n\left[\dfrac{4y}{\sigma^2(1-\rho^2)}\right], & y \geq 0 \\ \exp\left[\dfrac{2y}{\sigma^2(1-\rho)}\right] G_n^m\left[\dfrac{-4y}{\sigma^2(1-\rho^2)}\right], & y < 0. \end{cases} \tag{31}$$

B. The Polynomials $G_m^n(x)$

We list here several of the properties of these polynomials.

Definition:

$$G_m^n(x) \triangleq \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k}{k!} \qquad G_m^0(x) = e_m(x) \qquad G_0^n = 1. \tag{32}$$

Iterative relationships:

$$G_m^n(x) = \frac{x^m}{m!} + \sum_{r=0}^{n} G_{m-1}^r(x) = G_{m-1}^n(x) + G_m^{n-1}(x) \tag{33}$$

also,

$$G_m^n(x) = \sum_{\substack{k=0 \\ (m,n>p)}}^{p} \binom{p}{k} G_{m-k}^{n-p+k}(x) = \sum_{\substack{k=0 \\ (n \geq m)}}^{m} \binom{m}{k} G_k^{n-k}(x) = \sum_{\substack{k=0 \\ (m \geq n)}}^{n} \binom{n}{k} G_{m-k}^{k}(x). \tag{34}$$

Differentiation formula:

$$\frac{d^k}{dx^k} G_m^n(x) = G_{m-k}^n(x). \tag{35}$$

Addition formula:

$$G_m^n(x+y) = \sum_{k=0}^{m} \frac{y^k}{k!} G_{m-k}^n(x). \tag{36}$$

Relationships to Special Functions: The polynomials $G_m^n(x)$ may be related to the confluent hypergeometric functions $\Psi(\alpha,\beta;x)$ and ${}_2F_0(\gamma,\delta;\ ;-1/x)$, to the Laguerre polynomials $L_m^{\nu}(x)$, and to the Whittaker functions $W_{\lambda,\mu}(x)$ by the following formulas:

$$G_m^n(x) = \frac{1}{m!} \Psi(-m, -m-n; x) \tag{37}$$

$$= \frac{x^{m+n+1}}{m!} \Psi(n+1, m+n+2; x) \tag{38}$$

$$= \frac{x^m}{m!} {}_2F_0\left(-m, n+1;\ ;-\frac{1}{x}\right) \tag{39}$$

$$= \frac{1}{m!} e^{x/2} x^a W_{b,a+1/2}(x), \quad \text{for } a = \frac{m+n}{2},\ b = \frac{m-n}{2} \tag{40}$$

$$= (-1)^m L_m^{-m-n-1}(x). \tag{41}$$

By using the identity (40), our expression (31) for the pdf can be shown to be identical to that in [30, p. 63, eq. 28].

C. Equivalence of Forms

We now show that the form we have just derived for the probability density function is equivalent to that of (23) under the assumed conditions

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^m b^n}{m!\, n!} G_m^n(x)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{m} \binom{m+n-k}{n} \frac{x^k a^m b^n}{k!\, m!\, n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{m+n}{n} \frac{x^k a^{m+k} b^n}{k!\,(m+k)!\, n!}$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{a^{m+k} x^k}{(m+k)!\, k!} {}_1F_1(m+1, 1; b)$$

$$= \sum_{m=0}^{\infty} \left(\frac{a}{x}\right)^{(1/2)m} I_m(2\sqrt{ax})\, {}_1F_1(m+1, 1; b)$$

$$= e^b \sum_{m=0}^{\infty} \left(\frac{a}{x}\right)^{(1/2)m} I_m(2\sqrt{ax}) L_m(-b), \tag{42}$$

using Kummer's transformation on the last step.

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ON THE BINARY DPSK COMMUNICATION SYSTEMS IN CORRELATED GAUSSIAN NOISE

by

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This paper shows two new results on binary DPSK communication system. The paper is based on a probability density function developed under this contract.

ON THE BINARY DPSK COMMUNICATION SYSTEMS IN CORRELATED GAUSSIAN NOISE

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ABSTRACT

Using the recently developed probability density function we obtain "directly" the error rate expressions for the binary differential phase-shift keyed (DPSK) systems when the noise values at the sampling instants in adjacent time slots are statistically dependent. Two cases are considered: One corresponding to equal SNR at each of the two sampling instants, and the other to unequal SNRs. The consideration of the former case, together with the assumption of unequal *a priori* symbol probabilities P_0 and P_1 results in an error rate expression

$$P_E = \frac{1}{2}[1 + (P_0 - P_1)\rho]e^{-h^2}$$

where ρ is the noise correlation coefficient, and h^2 is the SNR. This expression shows clearly why P_E is independent of noise correlation when the source symbols are equi-probable provided intersymbol interference is assumed absent. We then obtain the error probability expression in terms of unequal SNRs (at the two sampling instants) and the correlation coefficient ρ. Since intersymbol interference in a binary DPSK system gives rise to unequal SNRs, this expression provides a useful formula for estimating the system performance under such circumstances.

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I. INTRODUCTION

The probability of error in binary differential phase-shift keyed (DPSK) systems has been analyzed by several authors [1]-[6]. Different methodologies have been employed by the previous authors in obtaining the error rate expression. In a broad sense, the previous techniques may be classed into two categories: "indirect" analogy technique and "characteristic function" method. By "indirect analogy technique" we mean that the computation of error probability is based on the knowledge of other communication systems such as binary orthogonal systems. In such an approach the technique does not require the probability density function (pdf) of the decision variable because the probability of error expression, namely the probability distribution function, can always be brought to a form that is equivalent to that of binary orthogonal system. Moreover, this technique has always been restricted to the case where the noise samples at adjacent time slots are assumed to be statistically independent. The characteristic function method [6], on the other hand, was employed to compute the error rate in a most general situation to account for the correlated noise and the intersymbol interference.

Our purpose here is to obtain the probability of error expression in a "direct" fashion by using the pdf of the decision variable developed recently by the authors [7]. We obtain the error rate expressions of the two cases of interest. The first is that corresponding to equal SNR at each of the two sampling instants, and the other to unequal SNRs. The consideration of the former with the assumption of unequal source symbol probabilities will lead to an error probability expression which shows clearly how it is related to the correlation between two noises. It also shows why the error probability is independent of noise correlation when the symbol probabilities are equiprobable provided that the intersymbol interference is assumed to be absent.

Finally, we consider the case of unequal SNRs at the two sampling instants. Hubbard [8] has shown that the effects of intersymbol interference is equivalent to consideration of different SNRs in the two received pulses. Thus, the error rate expression we obtain in terms of two different SNRs, along with the noise correlation coefficient, would serve a useful formula for estimating the "worst" [8] case intersymbol interference performance.

II. ANALYSIS

In binary DPSK systems, the sequence of binary source symbols 0 (space) and 1 (mark) differentially phase-modulate the carrier between, say, 0 and π radians. One possible rule of differential phase coding is that whenever the source symbol is 0 the carrier phase remains unchanged from that for the previous symbol (or bit), and if the source symbol is 1, the carrier phase is shifted π radians from that for the previous symbol. Thus, a change of π radians occurs in carrier phase whenever the symbol 1 is produced by the source.

The receiver model for the binary DPSK system to be considered is shown in Figure 1, which is an ideal product demodulator. As shown in the figure, the two inputs to the multiplier, $u_1(t)$ and $u_2(t)$, represent the received waveforms for two consecutive source symbols. Each input is assumed to contain bandpass information-carrying signals $S_i(t)$ and bandpass Gaussian noises $n_i(t)$, $i = 1,2$:

$$u_1(t) = S_1(t) + n_1(t) \tag{1}$$

$$u_2(t) = S_2(t) + n_2(t) \tag{2}$$

Let us assume that the "present" and the "preceding" source symbols are identified with carrier phases θ_1 and θ_2, respectively. Then we may write

$$S_1(t) = A \cos (\omega t-\theta_1) \tag{3}$$

and

$$S_2(t) = A \cos [\omega(t-T)-\theta_2] \tag{4}$$

If we assume the carrier frequency to have been chosen such that

$$\omega T = 2\pi k, \quad k \text{ integer,}$$

we have

$$S_2(t) = A \cos (\omega t-\theta_2) \tag{5}$$

The noises are bandpass stationary Gaussian processes and hence we may represent them in the form of well-known Rice decompositions [9],[10]:

$$n_1(t) = x_1(t) \cos \omega t + y_1(t) \sin \omega t \tag{6}$$

$$\begin{aligned} n_2(t) &= x_1(t-T) \cos [\omega(t-T)] + y_1(t-T) \sin [\omega(t-T)] \\ &= x_1(t-T) \cos \omega t + y_1(t-T) \sin \omega t \\ &\equiv x_2(t) \cos \omega t + y_2(t) \sin \omega t \end{aligned} \tag{7}$$

What is noted in (6) and (7) is that the noise processes $n_1(t)$ and $n_2(t)$ are not "two different" processes but rather $n_2(t)$ is a translated version of the noise process $n_1(t)$.

Our objective is to include in the analysis the influence of correlation between $n_1(t)$ and $n_2(t)$. Thus we have

$$E[n_1(t)n_2(t)] \equiv E[n_1(t)n_1(t-T)] = \sigma^2\rho(T) \tag{8}$$

and

$$\sigma^2 \triangleq E[n_i^2(t)] = E[x_i^2(t)] = E[y_i^2(t)]; \quad i=1,2, \tag{9}$$

and $\rho(T)$ is the normalized correlation coefficient of noises $n_1(t)$ and $n_2(t)$.

The pdf of the Decision Variable

The decision variable $y(t)$ at time t is the ideally lowpass filtered version of the product

$$x(t) = u_1(t)xu_2(t) = [S_1(t)+n_1(t)]x[S_2(t)+n_2(t)]$$

$$= [A\cos(\omega t-\theta_1)+n_1(t)]x[A\cos(\omega t-\theta_2)+n_2(t)]. \tag{10}$$

We now need to know the pdf of $y(t)$ at the zonal lowpass filter output (Figure 1). The problem of obtaining the pdf of the lowpass filter output for the system model that fits our situation was solved by Miller and Lee [7] in greater generality, including the case of correlated noises of unequal powers. Our need here is a special case of the problem treated in [7], and from (23) in [7] we have the pdf expressions as follows:

For $y > 0$ with noise correlation coefficient ρ:

$$p(y;\rho) = \frac{1}{\sigma^2} \exp\left[-\frac{2y}{\sigma^2(1+\rho)} - h_3^2 - \left(\frac{1-\rho}{2}\right) h_4^2\right]$$

$$\times \sum_{r=0}^{\infty} \left[\frac{h_3(1-\rho)\sqrt{1+\rho}}{\sqrt{8y/\sigma^2}}\right]^r I_r\left(\frac{2h_3}{\sigma}\sqrt{\frac{2y}{1+\rho}}\right) L_r\left[-\left(\frac{1+\rho}{2}\right) h_4^2\right] \quad (11a)$$

and for $y < 0$ with noise correlation coefficient ρ:

$$p(y;\rho) = \frac{1}{\sigma^2} \exp\left[\frac{2y}{\sigma^2(1-\rho)} - \left(\frac{1+\rho}{2}\right) h_3^2 - h_4^2\right]$$

$$\times \sum_{r=0}^{\infty} \left[\frac{h_4(1+\rho)\sqrt{1-\rho}}{\sqrt{-8y/\sigma^2}}\right]^r I_r\left(\frac{2h_4}{\sigma}\sqrt{\frac{-2y}{1-\rho}}\right) L_r\left[-\left(\frac{1-\rho}{2}\right) h_3^2\right] \quad (11b)$$

where

$$h_3^2 \triangleq \frac{1}{2(1+\rho)} [h_1^2+h_2^2+2h_1h_2 \cos(\theta_1-\theta_2)] \quad (12a)$$

$$h_4^2 \triangleq \frac{1}{2(1-\rho)} [h_1^2+h_2^2-2h_1h_2 \cos(\theta_1-\theta_2)] \quad (12b)$$

$I_r(\ )$ is the modified Bessel function of the first kind of order r

$L_r(\ )$ is the Laguerre polynomial of order r

and h_i^2 is the signal-to-noise ratio (SNR) in channel i, $i=1,2$.

It is evident that the pdf given in (11) is that of a general form in that the two inputs to the multiplier (Figure 1) have different signal power levels and the noises are correlated. In another sense, however, the pdf given in (11) is a "restricted" form in that the noise inputs are of equal power and their spectra are even about the center frequency of the band (symmetrical bandpass, see Section IV of [7]). With the above pdf we will be able to obtain the error probabilities of binary DPSK system under two different assumptions of practical interests. Namely, the error rates for equal and unequal input power level, respectively.

It has been known [5], [6] that when the source symbols are equi-probable and the intersymbol interference is assumed absent, the error probability of a binary DPSK system is independent of noise correlation. However, when the prior probabilities of the source sysmbols are unequal, the error probability is found to depend very much on the noise correlation. The absence of intersymbol interference corresponds to a situation where the two input powers to the multiplier are equal. It may be of interest to know, then, in what manner the error probability is affected by the noise correlation. In fact, in a laboratory simulation for DPSK system, the binary symbols are not always generated in an equi-probable fashion, and the simulation results are not always "poorer" than the theoretical prediction of the ideal situation. As will be noted subsequently, the simulation results that are "better" than the ideal theoretical predictions are readily reconcilable in view of the error rate expressions to be derived in the following section.

Also of interest is the error probability expression for the binary DPSK system for unequal input power levels. A practical situation which corresponds to this assumption is that of intersymbol interference, since the attribute of intersymbol interference is the unequal pulse levels for the two successive symbols. It will be explicit in the expression to be derived later that in such case the error performance depends upon the noise correlation regardless of the distribution of the source symbols.

III. THE PROBABILITY OF ERROR WHEN THE SUCCESSIVE PULSES HAVE EQUAL POWER

The pdf that is applicable to the particular situation where the two input channels to the multiplier (Figure 1) have equal SNRs is obtained when we substitute into (11) the conditions

$$h_1^2 = h_2^2 = h^2 = A^2/2\sigma^2. \tag{13}$$

We then obtain:

$$p(y;\rho) = \begin{cases} \dfrac{1}{\sigma^2} \exp\left\{-\dfrac{2y}{\sigma^2(1+\rho)} - \dfrac{2h^2}{1+\rho}\cos^2\left(\dfrac{\theta_1-\theta_2}{2}\right) - h^2 \sin^2\left(\dfrac{\theta_1-\theta_2}{2}\right)\right\} \\ \quad \times \displaystyle\sum_{r=0}^{\infty} \left[\dfrac{h(1-\rho)\cos\dfrac{(\theta_1-\theta_2)}{2}}{\sqrt{4y/\sigma^2}}\right]^r I_r\left[\dfrac{4h\cos\dfrac{(\theta_1-\theta_2)}{2}}{1+\rho}\sqrt{\dfrac{y}{\sigma^2}}\right] \\ \quad \times L_r\left[-h^2\sin^2\left(\dfrac{\theta_1-\theta_2}{2}\right)\left(\dfrac{1+\rho}{1-\rho}\right)\right]; \quad y > 0 & \text{(14a)} \\ \dfrac{1}{\sigma^2} \exp\left\{\dfrac{2y}{\sigma^2(1-\rho)} - h^2\cos^2\left(\dfrac{\theta_1-\theta_2}{2}\right) - \dfrac{2h^2}{1-\rho} \sin^2\left(\dfrac{\theta_1-\theta_2}{2}\right)\right\} \\ \quad \times \displaystyle\sum_{r=0}^{\infty} \left[\dfrac{h(1+\rho)\sin\left(\dfrac{\theta_1-\theta_2}{2}\right)}{\sqrt{-4y/\sigma^2}}\right]^r I_r\left[\dfrac{4h\sin\left(\dfrac{\theta_1-\theta_2}{2}\right)}{1-\rho}\sqrt{\dfrac{-y}{\sigma^2}}\right] \\ \quad \times L_r\left[-h^2\cos^2\left(\dfrac{\theta_1-\theta_2}{2}\right)\left(\dfrac{1-\rho}{1+\rho}\right)\right]; \quad y < 0 & \text{(14b)} \end{cases}$$

In binary DPSK system, the binary decision is based on the comparison of the decision variable with "zero" threshold. The reason is that when noises are assumed to be absent at the input, the decision variable is given by

$$y(t) = \frac{A^2}{2}\cos(\theta_1-\theta_2) \tag{15}$$

Since $\theta_1-\theta_2$ is either 0 or $\pm\pi$ in DPSK system, only the sign of $y(t)$ is significant in making a binary decision.

Let us assume that the reference (previous) phase θ_2 is 0. Then θ_1 is either 0 or π radians, depending upon whether the "next" symbol (present symbol) is 0 (space) or 1 (mark), so that $\theta_1-\theta_2 = 0$ or $\theta_1-\theta_2 = \pi$. If θ_2 is π instea then $\theta_1-\theta_2 = 0$ or $\theta_1-\theta_2 = -\pi$, again depending on whether "zero" or "one" is transmitted, respectively. From (14b) it can be easily verified that

$$p(y;\rho|\theta_1-\theta_2 = \pi) = p(y;\rho|\theta_1-\theta_2 = -\pi) \qquad (16)$$

Thus, the conditional pdf's are obtained as follows:

$$p(y;\rho|\text{space}) = p(y;\rho|\theta_1-\theta_2 = 0) \qquad (17a)$$

$$p(y;\rho|\text{mark}) = p(y;\rho|\theta_1-\theta_2 = \pi) \qquad (17b)$$

The density functions of (17) with the normalization of $u = y/\sigma^2$ are given by

$$p(u;\rho|\text{space}) = \begin{cases} \exp(\frac{2u}{1-\rho}) \exp(-h^2) & ; u<0 \\ \exp(- \frac{2u}{1+\rho})\exp(- \frac{2h^2}{1+\rho}) \sum_{r=0}^{\infty} \left[\frac{h(1-\rho)}{\sqrt{4u}}\right]^r I_r(\frac{4h}{1+\rho} \sqrt{u});u<0 \end{cases}$$

(18a)

and

$$p(u;\rho|\text{mark}) = \begin{cases} \exp(\frac{2u}{1-\rho})\exp(- \frac{2h^2}{1-\rho}) \sum_{r=0}^{\infty} \left[\frac{h(1+\rho)}{\sqrt{-4u}}\right]^r I_r(\frac{4h}{1-\rho} \sqrt{-u});u<0 \\ \exp(- \frac{2u}{1+\rho})\exp(-h^2) & ; u>0 \end{cases}$$

(18b)

From (18a) and (18b) we observe the "symmetry" properties to be

$$p(u;\rho|\text{space}) = p(-u;-\rho|\text{mark}) \tag{19}$$

The conditional pdf (19) is plotted under various parameter conditions in Figures 2-3. In Figure 2 the variations of the conditional pdf for different correlation coefficients are shown for fixed identical input SNRs of the two multiplier inputs. Note the dramatic influences of the correlation coefficients for each input SNR. In Figure 3 we have shown the variations of the pdf for different SNRs for fixed noise correlation coefficients.

To compute the probability of error we first obtain the conditional error probabilities:

$$P(e|\text{space}) = P_{rob}\{y<0|\theta_1-\theta_2 = 0\} \tag{20}$$

and

$$P(e|\text{mark}) = P_{rob}\{y>0|\theta_1-\theta_2 = \pm\pi\} \tag{21}$$

To compute (20), we need only obtain from (14)

$$p(y;\rho|\theta_1-\theta_2 = 0,\ y<0) \triangleq p^-(y;\rho|\theta_1-\theta_2 = 0)$$

which can be shown to be

$$p^-(y;\rho|\theta_1-\theta_2 = 0) = \frac{1}{\sigma^2} \exp\left[\frac{2y}{\sigma^2(1-\rho)} - 2h^2\left(\frac{1-\rho}{1-\rho^2}\right) + h^2\left(\frac{1-\rho}{1+\rho}\right)\right]$$

$$= \frac{1}{\sigma^2} \exp\left[\frac{2y}{\sigma^2(1-\rho)} - h^2\right] \tag{22}$$

In a similar manner, to compute (21) we need to obtain from (14)

$$p^+(y;\rho|\theta_1-\theta_2 = \pi) = \frac{1}{\sigma^2} \exp\left[-\frac{2y}{\sigma^2(1+\rho)} - 2h^2\left(\frac{1+\rho}{1-\rho^2}\right) + h^2\left(\frac{1+\rho}{1-\rho}\right)\right]$$

$$= \frac{1}{\sigma^2} \exp\left[-\frac{2y}{\sigma^2(1+\rho)} - h^2\right] \tag{23}$$

Using (22) and (23), we compute the conditional probabilities of error, and we get

$$P(e|\text{space}) = \int_{-\infty}^{0} p^{-}(y;\rho|\theta_1-\theta_2 = 0)\, dy = \frac{1-\rho}{2}\, e^{-h^2} \tag{24}$$

and

$$P(e|\text{mark}) = \int_{0}^{\infty} p^{+}(y;\rho|\theta_1-\theta_2 = \pi)\, dy = \frac{1+\rho}{2}\, e^{-h^2} \tag{25}$$

Assuming that the a priori probabilities of source symbol 0 (space) and 1 (mark) to be P_0 and P_1, respectively, we obtain the unconditional probability of error to be

$$\begin{aligned} P_E &= P_0 P(e|\text{space}) + P_1 P(e|\text{mark}) \\ &= \frac{1}{2}\,[1 + (P_1-P_0)\rho]\, e^{-h^2} \end{aligned} \tag{26}$$

When the source symbol probabilities are equiprobable, namely when $P_0 = P_1$, (26) reduces to the well known classical expression $P_E = \frac{1}{2}\exp(-h^2)$ for the binary DPSK error rate, where h^2 is the input carrier-to-noise power ratio (SNR).

From the above expressions, we observe the known fact [5],[6] that the error probability is independent of noise correlation when the source symbols are equi-probable. It is interesting to note the simplistic manner in which these variables affect the ideal classical error rate expression.

IV. THE PROBABILITY OF ERROR WHEN THE SUCCESSIVE PULSES HAVE UNEQUAL POWER

Our purpose here is to compute the probability of error in the binary DPSK system when the two multiplier inputs have different SNRs. As alluded to in the earlier discussions, this case corresponds to that of intersymbol interference. Our aim is to show that, under such conditions, the error rate expression is not independent of the noise correlation $\rho(T)$ even if the binary source symbols are assumed to be equi-probable.

Here we make use of the general form of the pdf given in (11). The method of analysis parallels that considered in Section III.

Let us assume that the reference phase θ_2 is 0 and the transmitted source symbol is 1 (mark). Then $\theta_1-\theta_2 = \pi$. The conditional probability of error is thus computed from

$$P(e;\rho,h_1,h_2|\text{mark}) = \int_0^\infty p^+(y;\rho,h_1,h_2|\theta_1-\theta_2 = \pi)dy \tag{27}$$

where the pdf in the integrand is given by (11a) with $\theta_1-\theta_2 = \pi$. The integration indicated by (27) then results in

$$P(e;\rho,h_1,h_2|\text{mark}) = \frac{1}{\sigma^2}\exp\left[-\frac{(h_1+h_2)^2}{4} - \frac{(h_1-h_2)^2}{2(1+\rho)}\right]$$
$$\times \sum_{r=0}^{\infty}\left[\frac{\frac{(1-\rho)}{\sqrt{2}}(h_1-h_2)}{\sqrt{8/\sigma^2}}\right]^r L_r\left[-\frac{1}{4}\frac{(1+\rho)}{(1-\rho)}(h_1+h_2)^2\right] F_r(\rho;h_1,h_2) \tag{28}$$

where

$$F_r(\rho;h_1,h_2) \triangleq \int_0^\infty e^{-\alpha y} y^{-r/2} I_r(2\beta\sqrt{y})dy \tag{29a}$$

and where

$$\alpha \triangleq \frac{2}{\alpha^2(1+\rho)} \tag{29b}$$

$$\beta \triangleq \frac{h_1-h_2}{\sigma(1+\rho)} \tag{29c}$$

Using the formulas 9.220.2 and 6.643.2 in [11], one can show that (29a) reduces to

$$F_r(\rho;h_1,h_2) = \frac{1}{r!}\beta^r \alpha^{-1}\ {}_1F_1(1,r+1;\beta^2/2\alpha) \tag{30}$$

where ${}_1F_1(\ )$ is the confluent hypergeometric function.

Putting (29b)-(30) into (28) we obtain

$$P(e;\rho,h_1,h_2|\text{mark}) = \frac{1+\rho}{2}\exp\left[-\frac{(h_1+h_2)^2}{4} - \frac{(h_1-h_2)^2}{2(1+\rho)}\right]$$

$$\times \sum_{r=0}^{\infty} \frac{1}{r!}\left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)(h_1-h_2)^2\right]^r {}_1F_1\left[1,r+1;\frac{(h_1-h_2)^2}{4(1+\rho)}\right]$$

$$\times L_r\left[-\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)(h_1+h_2)^2\right] \tag{31}$$

It can be shown that the other conditional probability of error is exactly the same as (31) except the sign change in ρ:

$$P(e;\rho,h_1,h_2|\text{space}) = P(e;-\rho,h_1,h_2|\text{mark}) \tag{32}$$

When $h_1=h_2$ is imposed on (31) and (32), the results reduce to the expression we have already obtained in Section III [see (24) and (25)]. That is, when $h_1^2 = h_2^2 = h^2$

$$P(e;\rho,h_1,h_2|\text{mark}) = \frac{1+\rho}{2}e^{-h^2} \tag{33a}$$

and

$$P(e;\rho,h_1,h_2|\text{space}) = \frac{1-\rho}{2}e^{-h^2} \tag{33b}$$

The probability of error for the case of unequal input SNRs for equi-probable binary DPSK system is then given by

$$P_E = \frac{1}{2}\left[P(e;\rho,h_1,h_2|\text{mark}) + P(e;\rho,h_1,h_2|\text{space})\right]$$

$$= \frac{1}{2}\left[P(e;\rho,h_1,h_2|\text{mark}) + P(e;-\rho,h_1,h_2|\text{mark})\right] \qquad (34)$$

where it is noted that the conditional error probability (31) is all that is needed to compute (34).

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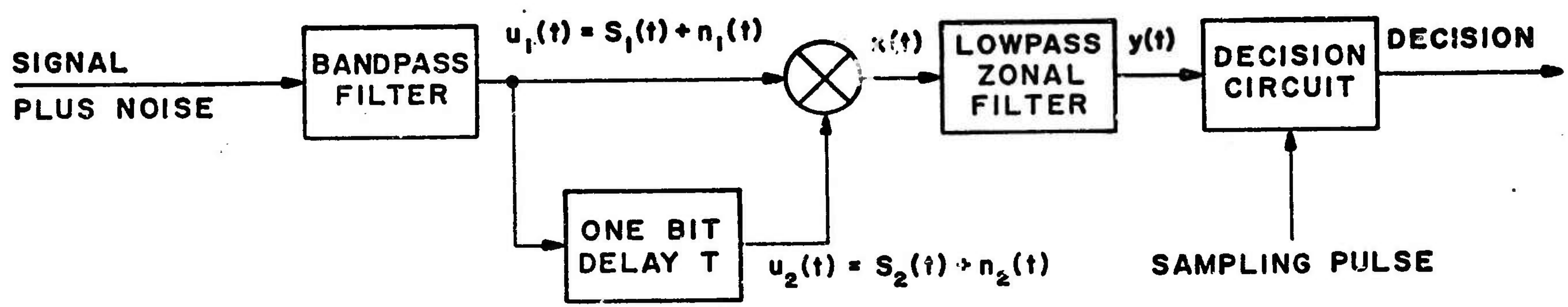

Figure 1: The receiver block diagram for binary DPSK communication system.

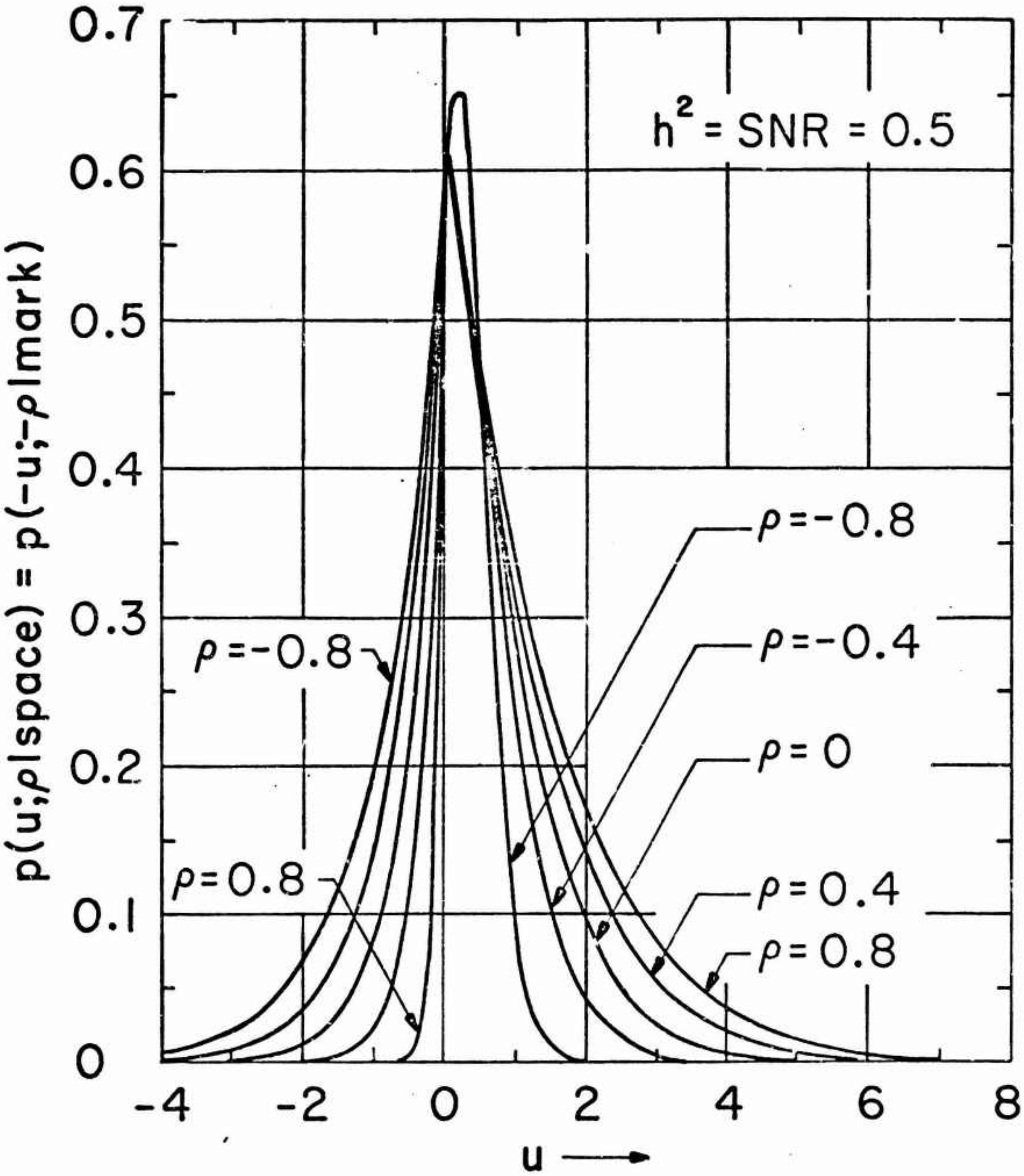

Figure 2a: pdf of the normalized ($u=y/\sigma^2$) decision variable (cross-correlator output) with noise correlation coefficients as parameters when h^2 = SNR = 0.5.

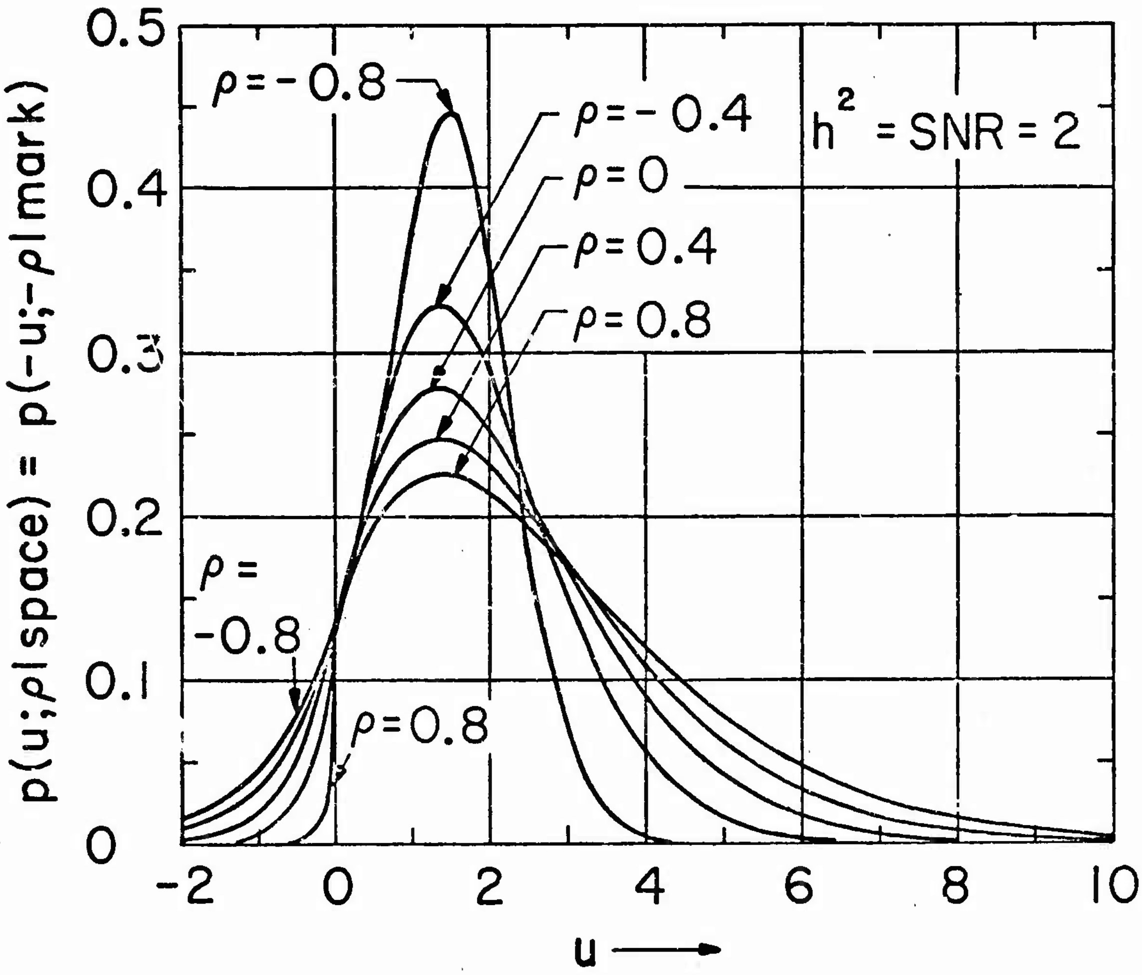

Figure 2b: pdf of the normalized ($u=y/\sigma^2$) decision variable (cross-correlator output) with noise correlation coefficients as parameters when h^2 = SNR = 2.

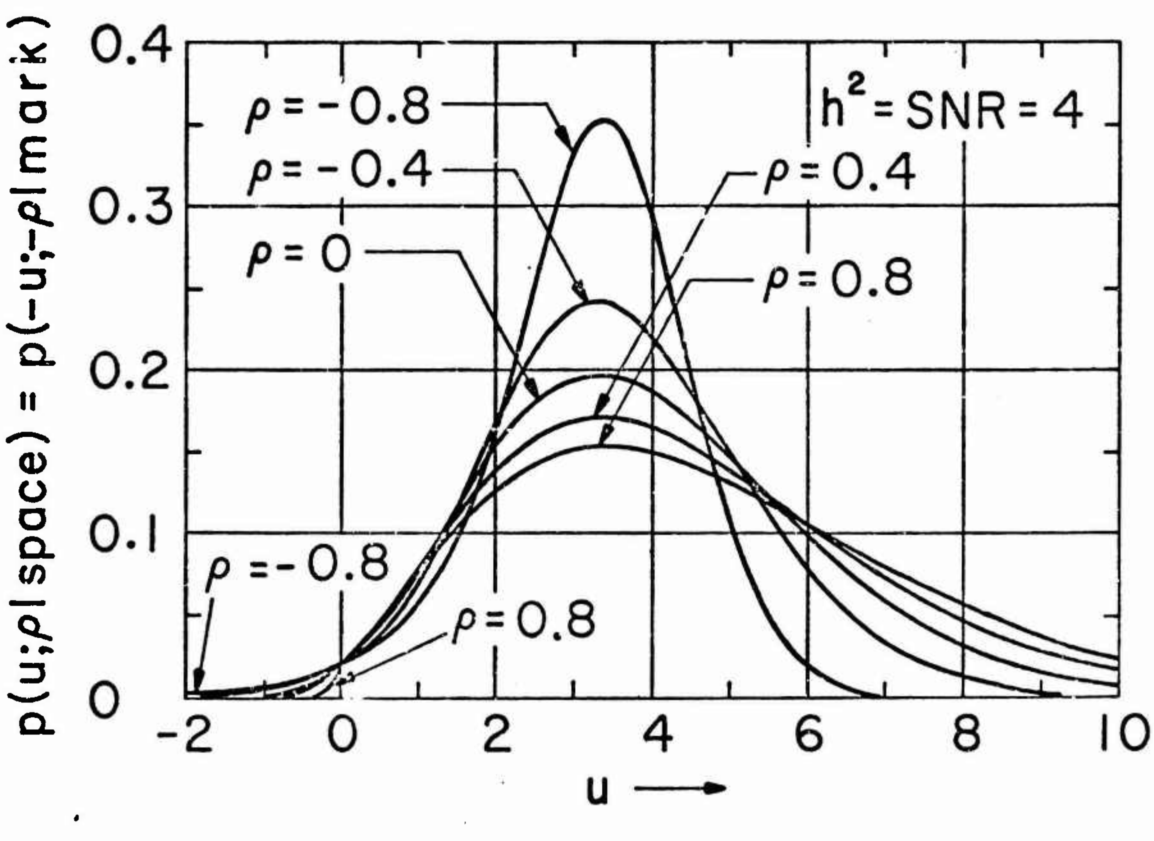

Figure 2c: pdf of the normalized ($u=y/\sigma^2$) decision variable (cross-correlator output) with noise correlation coefficients as parameters when h^2 = SNR = 4.

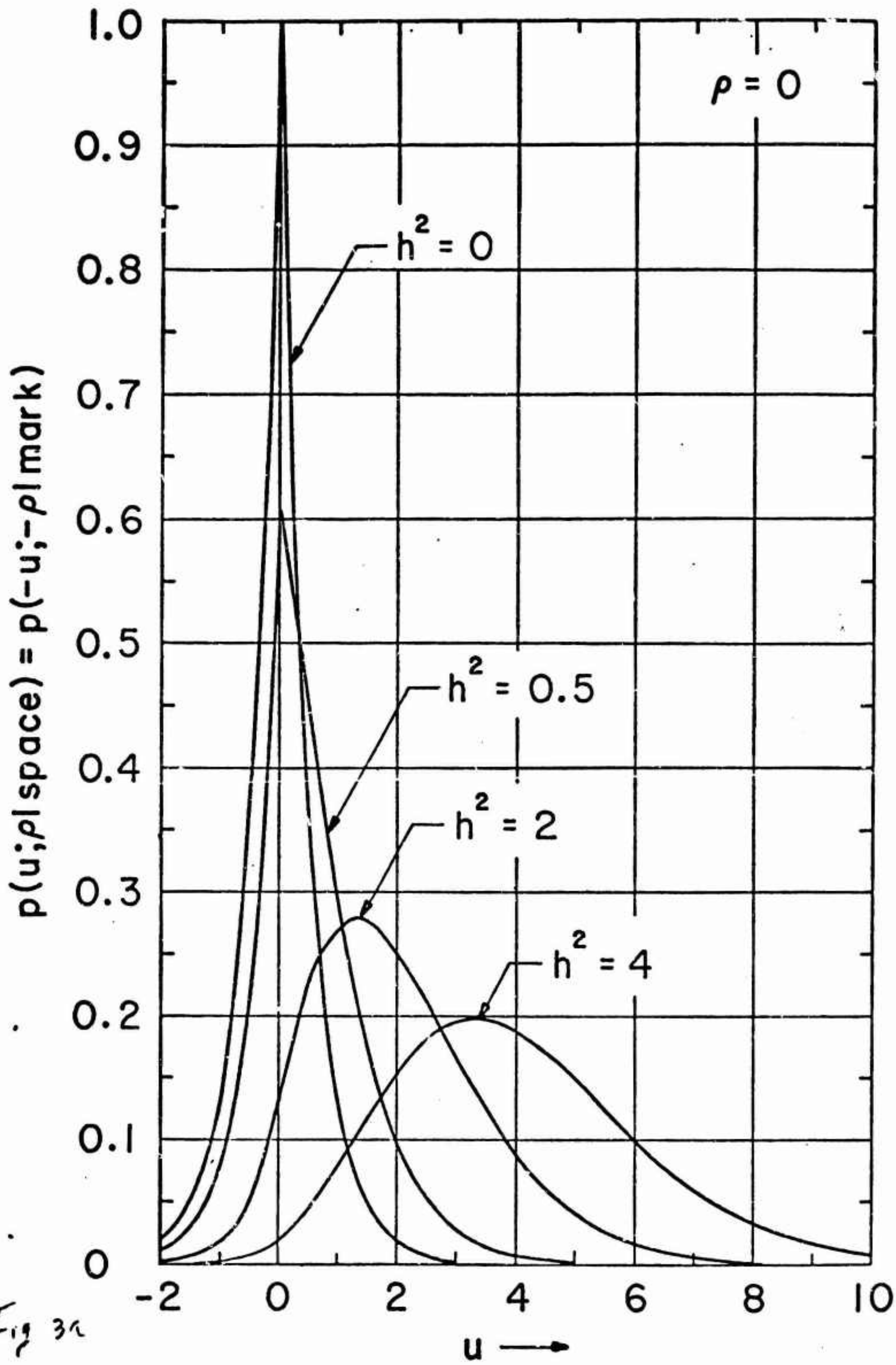

Fig 3a

Figure 3a: pdf of the normalized ($u=y/\sigma^2$) decision variable (cross-correlator output) with $h^2 \equiv$ SNR as parameters for $\rho = 0$ (a "classical" case).

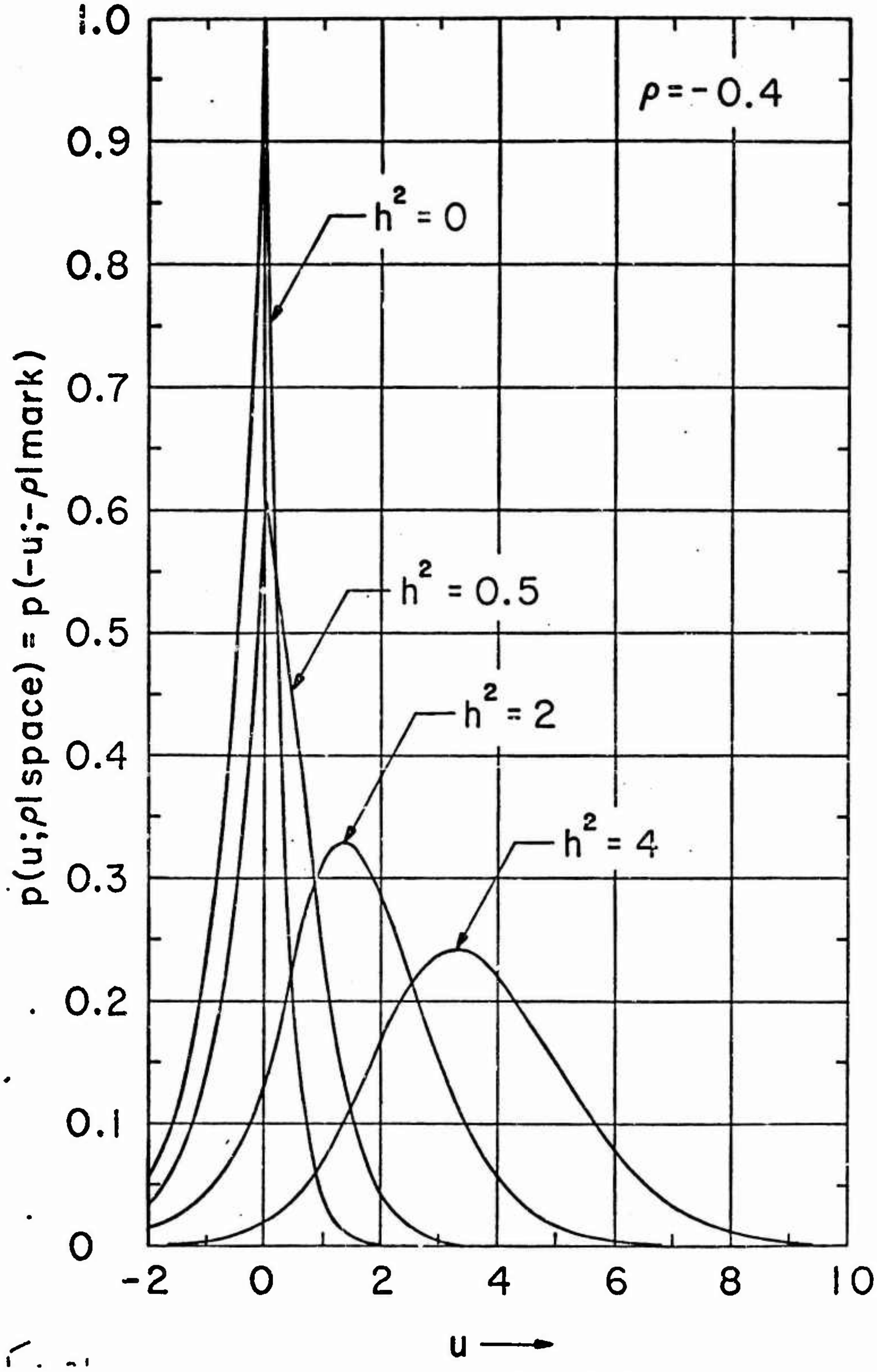

Figure 3b: pdf of the normalized ($u=y/\sigma^2$) decision variable (cross-correlator output) with $h^2 \equiv$ SNR as parameters for $\rho = -0.4$.

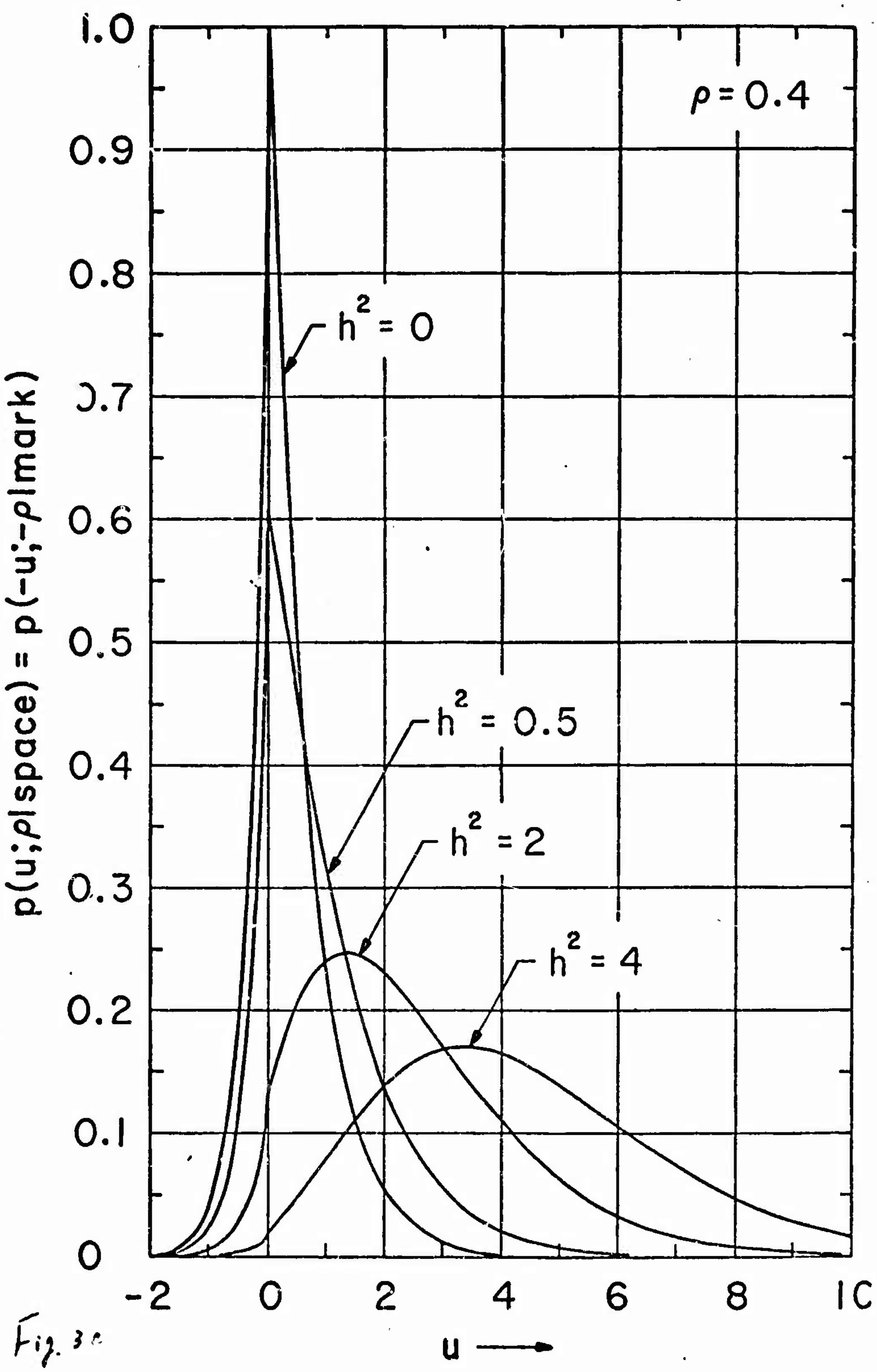

Figure 3c: pdf of the normalized ($u=y/\sigma^2$) decision variable (cross-correlator output) with $h^2 \equiv$ SNR as parameters for $\rho = 0.4$.